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PARABOLIC ANISOTROPIC PROBLEMS WITH LOWER ORDER TERMS AND INTEGRABLE DATA

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Abstract. In this paper we are concerned with the study of a class of second-order quasilinear parabolic equations involving Leray-Lions type operators with anisotropic growth conditions. By an approximation argument, we estabilsh the existence of entropy solutions in the framework of anisotropic parabolic Sobolev spaces when the initial condition and the data are assumed to be merely integrable. In addition, we prove that entropy solutions coincide with the renormalized solutions.

1. Introduction

Let Ω be an open bounded domain in $\mathbb{R}^N$, $N \geqslant 2$, let Q_T be the cylinder $\Omega \times (0,T)$ with $T > 0$ and $\Sigma_T = \partial\Omega \times (0,T)$. In framework of parabolic problems in isotropic case, Boccardo et al. studied in [16] the existence and regularity of solutions for the nonlinear parabolic equation

$$\begin{cases} u_t - \text{div}\,(|\nabla u|^{p-2}\nabla u) + \alpha_0 |u|^{s-1}u = f & \text{in} \quad Q_T, \\ u \; = 0 & \text{on} \quad \Sigma_T, \\ u(x,0) = 0 & \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

where $p > 1 + \frac{N}{N+1}$, $s > \frac{p(N+1)-N}{N}$, $\alpha_0 > 0$ and the data f is assumed to be in $L^1(Q_T)$. Other problems have been considered in this direction, see e.g. [6, 7, 14, 16].

Note that, the notion of renormalized solution has been introduced by DiPerna and Lions [20] in their study of the Boltzmann equation, then adapted to the study of elliptic problems with L^1-data by Boccardo, Giachetti, Diaz, and Murat [13] and Lions and Murat (see Lions book on the Navier-Stokes equations [29]). However the notion of entropy solutions was introduced independently by Bénilan et al. [5], see also [2].

In the last years, anisotropic Sobolev spaces have attracted a lot of attention, the impulse for this mainly comes from their applications of electro-rheological fluids and image processing (we refer the reader to [30, 31, 35] for more details).

In the context of stationary problems, let us mention here that several studies have been devoted to the investigation of related problems and a lot of papers have appeared

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dealing with equations involving anisotropic elliptic problems, see e.g. [15, 19, 27, 32]. There is also a large amount of literature for such problems, which we do not need to cite here since the reader may easily find such papers, we restrict ourselves to cite some work related to our topic. In [8], Bendahmane et al. used Hedberg-type's approximation to prove existence of distributional solutions in appropriate anisotropic function spaces for some strongly nonlinear boundary value problems associated with an anisotropic second-order operator where the data is assumed to be in the dual, see also [11]. Let us point out that more work in this direction can be found in [18] where the authors studied the existence and uniqueness results for strongly nonlinear anisotropic problems when the nonlinear lower-order term having natural growth with respect to the gradient and the data is regular or merely integrable using penalization methods, and in [17], where the authors studied some anisotropic elliptic problems of higher order in L^1, we refer the reader to [10, 12].

Also in the context of degenerate parabolic problems, it would be interesting to refer to the works [9, 22]. In [22], Elmahi et al. used an approximation theorem in inhomogeneous Orlicz–Sobolev spaces to solve a second-order parabolic equation in Orlicz spaces, see also [21, 25, 26] for related topics. As for the variable exponent case, in [9], the authors proved the existence and uniqueness of the renormalized solutions to the nonlinear parabolic problem involving the $p(x)$-Laplace in the framework of variable exponent Sobolev spaces

$$\begin{cases} u_t' - \operatorname{div}\,(|\nabla u|^{p(x)-2}\nabla u) = f & \text{in} \quad Q_T = \Omega\times(0,T),\\ u\ = 0 & \text{on} \quad \Sigma_T = \partial\Omega\times(0,T),\\ u(x,0) = u_0(x) & \text{in} \quad \Omega, \end{cases} \tag{1.2}$$

with $f \in L^1(\Omega)$. Moreover, they proved that $u \in L^{q(\cdot)}(0,T;W_0^{1,q(\cdot)}(\Omega))$ when $p(\cdot) > 2 - \frac{1}{N+1}$ for all continuous variable exponents $q(\cdot)$ on $\overline{\Omega}$ satisfying $1 \leqslant q(x) < \frac{N(p(x)-1)+p(x)}{N+1}$ for all $x \in \overline{\Omega}$.

Recently, Abdou et al. proved in [1] the existence of weak solutions for some quasilinear anisotropic parabolic problem when the data is assumed to be in the dual and the initial condition in $L^2(\Omega)$.

The aim of this paper is to establish the existence of entropy and renormalized solutions for some quasilinear anisotropic parabolic problem associated with a second-order operator of Leray-Lions type

$$\begin{cases} u_t - \sum_{i=1}^{N} D^i a_i(x,t,u,\nabla u) + g(x,t,u) = f & \text{in} \quad Q_T,\\ u(x,t) = 0 & \text{on} \quad \Sigma_T,\\ u(x,0) = u_0(x) & \text{in} \quad \Omega, \end{cases} \tag{1.3}$$

where $f \in L^1(Q_T)$, $u_0 \in L^1(\Omega)$, and the Carathéodory functions $a_i(x,s,\xi)$ satisfying some anisotropic growth conditions.

The main difficulty in proving the existence of solutions stems from the fact that the operator Au is not coercive in the anisotropic parabolic space $L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$, to overcome this difficulty, we will penalize the approximate problems.

This paper is organized as follows. In section 2 we present some definitions and results concerning the anisotropic parabolic spaces. We introduce in section 3 some essential assumptions on $a_i(x,t,s,\xi)$ and $g(x,t,s)$ to assure the existence of entropy solutions for our quasilinear anisotropic parabolic problem. Section 4 will be devoted to show the existence of entropy solutions for our problem (1.3). In the last section, we prove that the entropy solutions of the problem (1.3) are also renormalized solutions.

2. Preliminaries

In this section, we list briefly some definitions and well known facts about anisotropic Sobolev spaces used to study our quasilinear parabolic problem (1.3).

Let Ω be an open bounded domain in $\mathbb{R}^N$ $(N \geqslant 2)$ with boundary $\partial\Omega$, and let $p_0, p_1, \ldots, p_N$ be $N+1$ exponents, with $1 < p_i < \infty$ for $i = 1, \ldots, N$. We denote

$$\vec{p} = \{p_0, p_1, \ldots, p_N\}, \qquad D^0 u = u \quad \text{and} \quad D^i u = \frac{\partial u}{\partial x_i} \quad \text{for} \quad i = 1, \ldots, N, \tag{2.1}$$

and

$$\underline{p} = \min\{p_0, p_1, p_2, \ldots, p_N\} \qquad \text{then} \qquad \underline{p} > 1. \tag{2.2}$$

The anisotropic Sobolev space $W^{1,\vec{p}}(\Omega)$ is defined as follows

$$W^{1,\vec{p}}(\Omega) = \left\{u \in L^{p_0}(\Omega) \quad \text{and} \quad D^i u \in L^{p_i}(\Omega) \quad \text{for} \quad i = 1, 2, \ldots, N\right\}$$

endowed with the norm

$$\|u\|_{1,\vec{p}} = \sum_{i=0}^{N} \|D^i u\|_{L^{p_i}(\Omega)}. \tag{2.3}$$

The space $\left(W^{1,\vec{p}}(\Omega), \|u\|_{1,\vec{p}}\right)$ is a separable and reflexive Banach space (cf. [31, 33, 34]).

We define also $W_0^{1,\vec{p}}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,\vec{p}}(\Omega)$ with respect to the norm (2.3).

Let us recall the Poincaré and Sobolev type inequalities in the anisotropic Sobolev space.

PROPOSITION 2.1. *Let $u \in W_0^{1,\vec{p}}(\Omega)$, we have*
(i) *Poincaré inequality: there exists a constant $C_p > 0$ such that*

$$\|u\|_{L^{p_i}(\Omega)} \leqslant C_p \|D^i u\|_{L^{p_i}(\Omega)} \text{ for any } i = 1, \ldots, N.$$

(ii) *Sobolev inequality: there exists a constant $C_s > 0$ such that*

$$\|u\|_{L^q(\Omega)} \leqslant \frac{C_s}{N} \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}(\Omega)},$$

where

$$\frac{1}{\overline{p}} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{p_i} \qquad \text{and} \qquad \begin{cases} q = \overline{p}^* = \dfrac{N\overline{p}}{N - \overline{p}} & \text{if} \quad \overline{p} < N, \\ q \in [1, +\infty[& \text{if} \quad \overline{p} \geqslant N. \end{cases}$$

LEMMA 2.1. *Let Ω be a bounded open set in $\mathbb{R}^N$ $(N \geqslant 2)$, we set*

$$s = \max(q, \max_{1 \leqslant i \leqslant N} p_i),$$

then, we have the following embedding:

- *if $\overline{p} < N$ then the embedding $W_0^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^r(\Omega)$ is compact for any $r \in [1, s[$,*
- *if $\overline{p} = N$ then the embedding $W_0^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^r(\Omega)$ is compact for any $r \in [1, +\infty[$,*
- *if $\overline{p} > N$ then the embedding $W_0^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^\infty(\Omega) \cap C^0(\overline{\Omega})$ is compact.*

For more details, we refer the reader to [23].

PROPOSITION 2.2. *We denote the dual of the anisotropic Sobolev space $W_0^{1,\vec{p}}(\Omega)$ by $W^{-1,\vec{p}'}(\Omega)$, where $\vec{p}' = \{p'_0, p'_1, \dots, p'_N\}$ and $\frac{1}{p'_i} + \frac{1}{p_i} = 1$.*

For each $F \in W^{-1,\vec{p}'}(\Omega)$ there exists $F_i \in L^{p'_i}(\Omega)$ for $i = 0, 1, \dots, N$, such that $F = F_0 - \sum_{i=1}^{N} D^i F_i$. Moreover, for all $u \in W_0^{1,\vec{p}}(\Omega)$, we have

$$\langle F, u \rangle = \sum_{i=0}^{N} \int_\Omega F_i D^i u \, dx.$$

We define a norm on the dual space by

$$\|F\|_{-1,\vec{p}'} = \inf\Big\{ \sum_{i=0}^{N} \|F_i\|_{p'_i} \,/\, F = F_0 - \sum_{i=1}^{N} D^i F_i \ \text{ with } \ F_0 \in L^{s'}(\Omega) \ \text{ and } \ F_i \in L^{p'_i}(\Omega) \Big\}.$$

We refer to [8] for more details.

DEFINITION 2.1. Let $k > 0$, we consider the truncation function $T_k(\cdot) : \mathbb{R} \longmapsto \mathbb{R}$, given by

$$T_k(s) = \begin{cases} s & \text{if} \quad |s| \leqslant k, \\ k\dfrac{s}{|s|} & \text{if} \quad |s| > k. \end{cases}$$

We define

$$\mathscr{T}_0^{1,\vec{p}}(\Omega) := \{u : \Omega \mapsto \mathbb{R} \text{ measurable, such that } T_k(u) \in W_0^{1,\vec{p}}(\Omega) \text{ for any } k > 0\}.$$

PROPOSITION 2.3. *Let $u \in \mathscr{T}_0^{1,\vec{p}}(\Omega)$. For $i = 1, \dots, N$, there exists a unique measurable function $v_i : \Omega \mapsto \mathbb{R}$ such that*

$$D^i T_k(u) = v_i \cdot \chi_{\{|u| < k\}} \qquad \text{a.e. in} \quad \Omega, \quad \text{for any} \quad k > 0,$$

where χ_A denotes the characteristic function of a measurable set A. The functions v_i are called the weak partial derivatives of u and are still denoted $D^i u$. Moreover, if u belongs to $W_0^{1,1}(\Omega)$, then v_i coincides with the standard distributional derivative of u, that is $v_i = D^i u$.

2.1. Parabolic spaces

Let $Q_T = \Omega \times (0,T)$ with $0 < T < \infty$. We introduce the anisotropic parabolic space $L^{\vec{p}}(0,T;W^{1,\vec{p}}(\Omega))$ by

$$L^{\vec{p}}(0,T;W^{1,\vec{p}}(\Omega)) = \left\{ u \text{ measurable function } \ / \ \sum_{i=0}^{N} \int_0^T \|D^i u\|_{p_i}^{p_i}\, dt < \infty \right\}, \qquad (2.4)$$

endowed with the norm

$$\|u\|_{L^{\vec{p}}(0,T;W^{1,\vec{p}}(\Omega))} = \sum_{i=0}^{N} \|D^{\alpha} u\|_{L^{p_i}(Q_T)}.$$

We introduce the functional space $L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$, defined by

$$L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega)) = \left\{ u \in L^{\vec{p}}(0,T;W^{1,\vec{p}}(\Omega)) \ / \ u = 0 \text{ on } \partial\Omega \times [0,T] \right\}. \qquad (2.5)$$

The spaces $L^{\vec{p}}(0,T;W^{1,\vec{p}}(\Omega))$ and $L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$ are separable and reflexive Banach spaces.

The dual space of $L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$ is defined as follows

$$L^{\vec{p'}}(0,T;W^{-1,\vec{p'}}(\Omega)) = \left\{ F = F_0 - \sum_{i=1}^{N} D^i F_i, \ \text{ with } \ F_i \in L^{p'_i}(Q_T) \ \text{ for } \ i = 0,1,\ldots,N \right\} \qquad (2.6)$$

normed by

$$\|F\|_{L^{\vec{p'}}(0,T;W^{-1,\vec{p'}}(\Omega))} = \inf \left\{ \sum_{i=0}^{N} \|F_i\|_{L^{p'_i}(Q_T)} \ / \ F = F_0 - \sum_{i=1}^{N} D^i F_i \ \text{ with } \ F_0 \in L^{s'}(Q_T) \right.$$
$$\left. \text{and } \ F_i \in L^{p'_i}(Q_T) \right\}.$$

The duality of the spaces $L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$ and $L^{\vec{p'}}(0,T;W^{-1,\vec{p'}}(\Omega))$ is given by the relation

$$\langle F, v \rangle = \sum_{i=0}^{N} \int_{Q_T} f_i D^i v(x)\, dx \quad \text{for all} \quad v \in L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega)).$$

LEMMA 2.2. (cf. [36]) *Let $1 \leqslant p < \infty$ and $r = 1$, or $p = \infty$ and $r > 1$.*
Let X, B and Y be three Banach spaces such that

the embedding $X \hookrightarrow\hookrightarrow B$ is compact and the embedding $B \hookrightarrow Y$ is continuous.

Let $(u_n)_n$ be a bounded sequence in $L^p(0,T;X)$, with $\left(\frac{\partial u_n}{\partial t}\right)_n$ is bounded in $L^r(0,T;Y)$, Then, there exists $u \in L^p(0,T;B)$ such that, up to a subsequence, we have

$$u_n \longrightarrow u \qquad in \quad L^p(0,T;B).$$

i.e. the sequence $(u_n)_n$ is relatively compact in $L^p(0,T;B)$.

REMARK 2.1. Let $p=r=1$, taking

$$X=W_0^{1,1}(\Omega), \qquad B=L^1(\Omega) \qquad \text{and} \qquad Y=W^{-1,1}(\Omega).$$

We have the embedding $W_0^{1,1}(\Omega)\hookrightarrow\hookrightarrow L^1(\Omega)$ is compact and the embedding $L^1(\Omega)\hookrightarrow W^{-1,1}(\Omega)$ is continuous. Then

$$\Big\{u\ :\ u\in L^1(0,T;W_0^{1,1}(\Omega)) \quad \text{and} \quad \frac{\partial u}{\partial t}\in L^1(0,T;W^{-1,1}(\Omega))\Big\}\hookrightarrow\hookrightarrow L^1(Q_T). \tag{2.7}$$

Moreover, since $\underline{p}>1,\ W_0^{1,\vec{p}}(\Omega)\hookrightarrow W_0^{1,1}(\Omega)$ and $W^{-1,\vec{p}'}(\Omega)\hookrightarrow W^{-1,1}(\Omega)$ are continuous, it follows that

$$\Big\{u\ :\ u\in L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega)) \quad \text{and} \quad \frac{\partial u}{\partial t}\in L^{\vec{p}'}(0,T;W^{-1,\vec{p}'}(\Omega))\Big\}\hookrightarrow\hookrightarrow L^1(Q_T). \tag{2.8}$$

3. Essential assumptions and some technical Lemmas

Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N\geqslant 2)$ with a Lipschitz boundary denoted by $\partial\Omega$. For $T>0$, we denote by Q_T the cylinder $\Omega\times(0,T)$ and by Σ_T the lateral surface $\partial\Omega\times(0,T)$.

We assume that the vector $\vec{p}=(p_0,p_1,\dots,p_N)$ satisfies $1<p_i<\infty$ for $i=0,1,\dots,N$, and

$$p_0\geqslant\max\{p_i, \quad i=1,2,\dots,N\}. \tag{3.1}$$

We consider a Leray-Lions operator A acted from $L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$ into its dual $L^{\vec{p}'}(0,T;W^{-1,\vec{p}'}(\Omega))$ defined by

$$Au=-\sum_{i=1}^N D^i a_i(x,t,u,\nabla u),$$

where $a_i(x,t,s,\xi)$ are Carathéodory functions (measurable with respect to (x,t) in Q_T for any (s,ξ) in $\mathbb{R}\times\mathbb{R}^N$, and continuous with respect to (s,ξ) in $\mathbb{R}\times\mathbb{R}^N$ for almost every (x,t) in Q_T) satisfying the following conditions

$$|a_i(x,t,s,\xi)|\leqslant\beta(K_i(x,t)+|s|^{p_i-1}+|\xi_i|^{p_i-1}), \tag{3.2}$$

$$a_i(x,t,s,\xi)\xi_i\geqslant\alpha|\xi_i|^{p_i}, \tag{3.3}$$

for any $\xi=(\xi_1,\dots,\xi_N)$ and $\xi'=(\xi'_1,\dots,\xi'_N)$, we have

$$(a_i(x,t,s,\xi)-a_i(x,t,s,\xi'))(\xi_i-\xi'_i)>0 \quad \text{for} \quad \xi_i\neq\xi'_i, \tag{3.4}$$

for a.e. $(x,t)\in Q_T$, and all $(s,\xi)\in\mathbb{R}\times\mathbb{R}^N$, where $K_i(x,t)$ is a nonnegative function lying in $L^{p'_i}(Q_T)$ and $\alpha,\beta>0$.

The nonlinear term g is a Carathéodory function which satisfies

$$|g(x,t,s)| \leqslant b(x,t) + \gamma |s|^{p_0-1}, \tag{3.5}$$

$$g(x,t,s)s \geqslant 0, \tag{3.6}$$

for a.e. $(x,t) \in Q_T$ and any $s \in \mathbb{R}$, the function $b : \Omega \times (0,T) \longmapsto \mathbb{R}^+$ with $b \in L^{p_0'}(Q_T)$, and $\gamma > 0$.

We consider the quasilinear anisotropic parabolic problem

$$\begin{cases} u_t - \sum_{i=1}^{N} D^i a(x,t,u,\nabla u) + g(x,t,u) = f & \text{in } Q_T, \\ u = 0 & \text{on } \Sigma_T, \\ u(x,0) = u_0 & \text{in } \Omega, \end{cases} \tag{3.7}$$

with $f \in L^1(Q_T)$ and $u_0 \in L^1(\Omega)$.

REMARK 3.1. The assumption (3.1) is essential to ensure that $a_i(x,t,u,\nabla u)$ belongs to $L^{p_i'}(Q_T)$. In the case of $Au = -\sum_{i=1}^{N} D^i a_i(x,t,\nabla u)$, the existence of an entropy solution is guaranteed, without using this assumption.

LEMMA 3.1. (see [24], Theorem 13.47) *Let $(u_n)_n$ be a sequence in $L^1(\Omega)$ and $u \in L^1(\Omega)$ such that*

(i) $u_n \to u$ *a.e. in* Ω,

(ii) $u_n \geqslant 0$ *and* $u \geqslant 0$ *a.e. in* Ω,

(iii) $\int_\Omega u_n \, dx \to \int_\Omega u \, dx$,

then $u_n \to u$ in $L^1(\Omega)$.

LEMMA 3.2. *Assuming that (3.2)–(3.4) hold. Let $(u_n)_n$ be a sequence in $L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$ such that $((u_n)_t)_n$ is bounded in $L^{\vec{p}'}(0,T;W^{-1,\vec{p}'}(\Omega))$, with $u_n \rightharpoonup u$ in $L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$ and*

$$\begin{aligned} \sum_{i=1}^{N} \int_{Q_T} \big(a_i(x,t,u_n,\nabla u_n) - a_i(x,t,u_n,\nabla u)\big)(D^i u_n - D^i u) \, dx \, dt \\ + \int_{Q_T} \Big(|u_n|^{p_0-2} u_n - |u|^{p_0-2} u\Big)(u_n - u) \, dx \, dt \longrightarrow 0, \end{aligned} \tag{3.8}$$

then $u_n \to u$ in $L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$ for a subsequence.

Proof. Let

$$D_n(x,t) = \sum_{i=1}^{N} \big(a_i(x,t,u_n,\nabla u_n) - a_i(x,t,u_n,\nabla u)\big)(D^i u_n - D^i u) \\ + \big(|u_n|^{p_0-2}u_n - |u|^{p_0-2}u\big)(u_n - u).$$

Thanks to (3.4), we have $D_n(x,t)$ is a positive function, and in view of (3.8), we get $D_n \to 0$ in $L^1(Q_T)$ as $n \to \infty$.

We also have $u_n \rightharpoonup u$ in $L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$, and in view of the compact embedding (2.8), we get $u_n \to u$ strongly in $L^1(Q_T)$. It follows that $u_n \to u$ a.e in Q_T, and since $D_n \to 0$ a.e in Q_T, there exists a subset B in Q_T with measure zero such that $\forall (x,t) \in Q_T \backslash B$ $|u_n(x,t)| < \infty$, $|D^i u_n(x,t)| < \infty$, $|K(x,t)| < \infty$, $u_n \to u$ and $D_n \to 0$ a.e. in Q_T. Then we have

$$\begin{aligned} D_n(x,t) &= \sum_{i=1}^{N} \big(a_i(x,t,u_n,\nabla u_n) - a_i(x,t,u_n,\nabla u)\big)(D^i u_n - D^i u) \\ &\quad + \big(|u_n|^{p_0-2}u_n - |u|^{p_0-2}u\big)(u_n - u) \\ &\geqslant \alpha \sum_{i=1}^{N} |D^i u_n|^{p_i} + \alpha \sum_{i=1}^{N} |D^i u|^{p_i} + |u_n|^{p_0} + |u|^{p_0} \\ &\quad - \beta \sum_{i=1}^{N} \big(K_i(x,t) + |u_n|^{p_i-1} + |D^i u_n|^{p_i-1}\big)|D^i u| \\ &\quad - \beta \sum_{i=1}^{N} \big(K_i(x,t) + |u_n|^{p_i-1} + |D^i u|^{p_i-1}\big)|D^i u_n| - |u_n|^{p_0-1}|u| - |u|^{p_0-1}|u_n| \\ &\geqslant \underline{\alpha} \sum_{i=0}^{N} |D^i u_n|^{p_i} - C_{x,t} \sum_{i=0}^{N} \Big(1 + |D^i u_n|^{p_i-1} + |D^i u_n|\Big), \end{aligned}$$

where $\underline{\alpha} = \min(1,\alpha)$ and $C_{x,t}$ is a constant depending on (x,t), without dependence on n, (since $u_n(x,t) \to u(x,t)$ then $(u_n)_n$ is bounded). Hence we obtain

$$D_n(x) \geqslant \sum_{i=0}^{N} |D^i u_n|^{p_i} \Big(\underline{\alpha} - \frac{C_{x,t}}{|D^i u_n|^{p_i}} - \frac{C_{x,t}}{|D^i u_n|} - \frac{C_{x,t}}{|D^i u_n|^{p_i-1}}\Big).$$

By a standard argument, the sequence $(D^i u_n)_n$ is bounded almost everywhere in Q_T. Indeed, if $|D^i u_n| \to \infty$ in a measurable subset $E \in Q_T$ then

$$\lim_{n\to\infty} \int_{Q_T} D_n(x)\,dx\,dt \geqslant \lim_{n\to\infty} \sum_{i=0}^{N} \int_E |D^i u_n|^{p_i} \big(\underline{\alpha} - \frac{C_{x,t}}{|D^i u_n|^{p_i}} - \frac{C_{x,t}}{|D^i u_n|} - \frac{C_{x,t}}{|D^i u_n|^{p_i-1}}\big)\,dx\,dt = \infty,$$

which is absurd since $D_n \to 0$ in $L^1(Q_T)$. Let now ξ_i^* be an accumulation point of $(D^i u_n)_n$, we have $|\xi_i^*| < \infty$ and by the continuity of the Carathéodory function $a(x,t,\cdot,\cdot)$, we obtain

$$\Big(a_i(x,t,u,\xi^*) - a_i(x,t,u,\nabla u)\Big)(\xi_i^* - D^i u) = 0.$$

Thanks to (3.4), we have $\xi_i^* = D^i u$, and the uniqueness of the accumulation point implies that $D^i u_n(x,t) \to D^i u(x,t)$ a.e in Q_T for $i = 0, 1, \cdots, N$.

Since $(a_i(x,t,u_n,\nabla u_n))_n$ is bounded in $L^{p_i'}(Q_T)$ and $a_i(x,t,u_n,\nabla u_n) \to a_i(x,t,u,\nabla u)$ a.e in Q_T, we can establish that

$$a_i(x,t,u_n,\nabla u_n) \rightharpoonup a_i(x,t,u,\nabla u) \quad \text{in} \quad L^{p_i'}(Q_T).$$

Using (3.8) and Lemma 3.1, we deduce that

$$|u_n|^{p_i} \longrightarrow |u|^{p_i} \quad \text{in} \quad L^1(Q_T), \tag{3.9}$$

and

$$a_i(x,t,u_n,\nabla u_n)D^i u_n \longrightarrow a_i(x,t,u,\nabla u)D^i u \quad \text{in} \quad L^1(Q_T). \tag{3.10}$$

According to the condition (3.3) , we have

$$\alpha |D^i u_n|^{p_i} \leqslant a_i(x,t,u_n,\nabla u_n)D^i u_n \quad \text{for} \quad i = 1, \ldots, N.$$

Let

$$y_n^i = \frac{a_i(x,t,u_n,\nabla u_n)D^i u_n}{\alpha} \qquad \text{and} \qquad y^i = \frac{a_i(x,t,u,\nabla u)D^i u}{\alpha}.$$

In view of Fatou's Lemma, we get

$$\int_{Q_T} 2y^i \, dx\, dt \leqslant \liminf_{n\to\infty} \int_{Q_T} (y_n^i + y^i - \frac{1}{2^{p_i-1}} |D^i u_n - D^i u|^{p_i}) \, dx\, dt.$$

Then $0 \leqslant -\limsup_{n\to\infty} \int_{Q_T} |D^i u_n - D^i u|^{p_i} \, dx\, dt$, and since

$$0 \leqslant \liminf_{n\to\infty} \int_{Q_T} |D^i u_n - D^i u|^{p_i} \, dx\, dt \leqslant \limsup_{n\to\infty} \int_{Q_T} |D^i u_n - D^i u|^{p_i} \, dx\, dt \leqslant 0.$$

It follows that $\int_{Q_T} |D^i u_n - D^i u|^{p_i} \, dx\, dt \longrightarrow 0$ as $n \to \infty$. Hence we get

$$D^i u_n \longrightarrow D^i u \quad \text{in} \quad L^{p_i}(Q_T) \quad \text{for} \quad i = 1, \ldots, N.$$

Finally, thanks to (3.9), we deduce that

$$u_n \longrightarrow u \qquad \text{in} \quad L^{\vec{p}}(0,T; W_0^{1,\vec{p}}(\Omega)).$$

This completes the proof. □

4. Main result

We set

$$\varphi_k(r) = \int_0^r T_k(s)\, ds = \begin{cases} \frac{r^2}{2} & \text{if} \quad |r| \leqslant k, \\ k|r| - \frac{k^2}{2} & \text{if} \quad |r| > k. \end{cases}$$

DEFINITION 4.1. Let $f \in L^1(Q_T)$ and $u_0 \in L^1(\Omega)$. A measurable function u is an entropy solution of the anisotropic parabolic problem (3.7), if $T_k(u) \in L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$ for all $k > 0$, and

$$\begin{cases} \int_\Omega \varphi_k(u-\psi)(T)\,dx - \int_\Omega \varphi_k(u-\psi)(0)\,dx + \int_{Q_T} \frac{\partial \psi}{\partial t} T_k(u-\psi)\,dx\,dt \\ \quad + \sum_{i=1}^N \int_{Q_T} a_i(x,t,u,\nabla u) D^i T_k(u-\psi)\,dx\,dt + \int_{Q_T} g(x,t,u) T_k(u-\psi)\,dx\,dt \\ \leqslant \int_{Q_T} f T_k(u-\psi)\,dx\,dt, \end{cases} \tag{4.1}$$

for any $\psi \in L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega)) \cap L^\infty(Q_T)$ with $\frac{\partial \psi}{\partial t} \in L^{\vec{p}'}(0,T;W^{-1,\vec{p}'}(\Omega)) + L^1(Q_T)$.

THEOREM 4.1. *Let $f \in L^1(Q_T)$ and $u_0 \in L^1(\Omega)$. Assuming that the assumptions (3.1)–(3.6) hold true. Then the quasilinear anisotropic parabolic problem (3.7) has at least one entropy solution.*

Proof.

Step 1: Approximate problem. Let $(f_n)_n$ be a sequence in $L^{\vec{p}'}(0,T;W^{-1,,\vec{p}'}(\Omega)) \cap L^1(Q_T)$ such that $f_n \to f$ in $L^1(Q_T)$, with $|f_n| \leqslant |f|$ (for example $f_n = T_n(f)$), and let $(u_{0,n})$ be a sequence in $C_0^\infty(\Omega)$ such that $u_{0,n} \to u_0$ in $L^1(\Omega)$ and $|u_{0,n}| \leqslant |u_0|$.

We consider the approximate problem

$$\begin{cases} \frac{\partial u_n}{\partial t} + A_n u_n + g_n(x,t,u_n) + \frac{1}{n}|u_n|^{p_0-2}u_n = f_n & \text{in} \quad \Omega \times (0,T), \\ u_k(x,t) = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ u_k(x,0) = u_0 & \text{in} \quad \Omega, \end{cases} \tag{4.2}$$

with $A_n u = -\sum_{i=1}^N D^i a_i(x,t,T_n(u),\nabla u)$ and $g_n(x,t,s) = \dfrac{g(x,t,s)}{1+\frac{1}{n}|g(x,t,s)|}$. Note that

$$g_n(x,t,s)s \geqslant 0, \qquad |g_n(x,t,s)| \leqslant |g(x,t,s)| \quad \text{and} \quad |g_n(x,t,s)| \leqslant n \quad \forall n \in \mathbb{N}^*.$$

We define the operator $G_n : L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega)) \longmapsto L^{\vec{p}'}(0,T;W^{-1,,\vec{p}'}(\Omega))$ by

$$\int_0^T \langle G_n u, v\rangle\,dt = \int_{Q_T} g_n(x,t,u)v\,dx\,dt + \frac{1}{n}\int_{Q_T} |u|^{p_0-2}u\,v\,dx\,dt \quad \forall v \in L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega)).$$

Thanks to (3.2), (3.5) and Hölder inequality, we have for all $u, v \in L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$

$$\begin{aligned} \left| \int_0^T \langle A_n u, v\rangle\,dt \right| &= \sum_{i=1}^N \int_{Q_T} |a_i(x,t,T_n(u),\nabla u)|\,|D^i v|\,dx\,dt \\ &\leqslant \sum_{i=1}^N \int_{Q_T} \beta(K_i(x,t) + |T_n(u)|^{p_i-1} + |D^i u|^{p_i-1})|D^i v|\,dx\,dt \\ &\leqslant \beta \sum_{i=1}^N (\|K_i(x,t)\|_{L^{p_i'}(Q_T)} + \|T_n(u)\|_{L^{p_i}(Q_T)}^{p_i-1} + \|D^i u\|_{L^{p_i}(Q_T)}^{p_i-1})\|D^i v\|_{L^{p_i}(Q_T)} \end{aligned} \tag{4.3}$$

and

$$\left|\int_0^T \langle G_n u, v\rangle\, dt\right| \leqslant \int_{Q_T} |g_n(x,t,u)|\,|v|\,dx\,dt + \frac{1}{n}\int_{Q_T} |u|^{p_0-1}\,|v|\,dx\,dt$$
$$\leqslant \int_{Q_T} \Big(\Big(\frac{1}{n}+1\Big)|u|^{p_0-1} + b(x,t)\Big)|v|\,dx\,dt \tag{4.4}$$
$$\leqslant \Big(\Big(\frac{1}{n}+1\Big)\|u\|^{p_0-1}_{L^{p_0}(Q_T)} + \|b(x,t)\|_{L^{p_0'}(Q_T)}\Big)\|v\|_{L^{p_0}(Q_T)}. \quad \square$$

LEMMA 4.1. *The operator* $B_n = A_n + G_n$ *is pseudo-monotone from* $L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$ *into* $L^{\vec{p}'}(0,T;W^{-1,\vec{p}'}(\Omega))$. *Moreover,* B_n *is coercive in the following sense*

$$\frac{\int_0^T \langle B_n v, v\rangle\, dt}{\|v\|_{L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))}} \longrightarrow \infty \qquad \textit{as} \quad \|v\|_{L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))} \to \infty.$$

Proof. In view of the inequality (4.3) and (4.4), the operator B_n is bounded. For the coercivity, thanks to (3.3) and (3.6), we have for all $u \in L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$

$$\begin{aligned}\int_0^T \langle B_n u, u\rangle\, dt &= \sum_{i=1}^N \int_{Q_T} a_i(x,t,T_n(u)\nabla u) D^i u\, dx\, dt + \int_{Q_T} g(x,t,u) u\, dx\, dt\\ &\quad + \frac{1}{n}\int_{Q_T} |u|^{p_0}\, dx\, dt\\ &\geqslant \alpha \sum_{i=1}^N \int_{Q_T} |D^i u|^{p_i}\, dx\, dt + \frac{1}{n}\int_{Q_T} |u|^{p_0}\, dx\, dt\\ &\geqslant \alpha \sum_{i=1}^N (\|D^i u\|^{\underline{p}}_{L^{p_i}(Q_T)} - 1) + \frac{1}{n}(\|u\|^{\underline{p}}_{L^{p_0}(Q_T)} - 1)\\ &\geqslant \alpha' \|v\|^{\underline{p}}_{L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))} - \alpha N - \frac{1}{n},\end{aligned}$$

with $\alpha' = \dfrac{\min(\alpha, \frac{1}{n})}{(N+1)^{\underline{p}-1}}$. It follows that

$$\frac{\int_0^T \langle B_n u, u\rangle\, dt}{\|u\|_{L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))}} \longrightarrow \infty \qquad \text{as} \quad \|u\|_{L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))} \to \infty.$$

It remain to show that B_n is pseudo-monotone. Let $(u_k)_k$ be a sequence in $L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$ such that

$$\begin{cases} u_k \rightharpoonup u & \text{in } L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega)),\\ B_n u_k \rightharpoonup \chi_n & \text{in } L^{\vec{p}'}(0,T;W^{-1,\vec{p}'}(\Omega)),\\ \limsup\limits_{k\to\infty} \displaystyle\int_0^T \langle B_n u_k, u_k\rangle\, dt \leqslant \int_0^T \langle \chi_n, u\rangle\, dt. \end{cases} \tag{4.5}$$

We prove that

$$\chi_n = B_n u \quad \text{and} \quad \int_0^T \langle B_n u_k, u_k \rangle \, dt \longrightarrow \int_0^T \langle \chi_n, u \rangle \, dt \ \text{ as } \ k \to +\infty.$$

First, we have $u_k \rightharpoonup u$ in $L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$ and in view of the compact embedding (2.8), we obtain $u_n \to u$ strongly in $L^1(Q_T)$, and a.e. in Q_T.

We have $(u_k)_k$ is a bounded sequence in $L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$. Using the growth condition (3.3), the Carathéodory function $(a_i(x,t,T_n(u_k),\nabla u_k))_k$ is bounded in $L^{p_i'}(Q_T)$. Therefore there exists a function $\varphi_i \in L^{p_i'}(Q_T)$ such that

$$a_i(x,t,T_n(u_k),\nabla u_k) \rightharpoonup \varphi_i \ \text{ in } \quad L^{p_i'}(Q_T) \quad \text{ for } i = 1,\dots,N. \tag{4.6}$$

Further, we have $g_n(x,t,u_k) \to g_n(x,t,u)$ a.e. in Q_T and $g_n(x,t,u_k) \leqslant n \in L^{p_0'}(Q_T)$. Hence, in view of Lebesgue's dominated convergence theorem, we deduce that

$$g_n(x,t,u_k) \longrightarrow g_n(x,t,u) \quad \text{ in } \quad L^{p_0'}(Q_T). \tag{4.7}$$

We also have

$$\frac{1}{n}|u_k|^{p_0-2}u_k \rightharpoonup \frac{1}{n}|u|^{p_0-2}u \quad \text{ in } \quad L^{p_0'}(Q_T). \tag{4.8}$$

Thus, for any $v \in L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$, we get

$$\begin{aligned}
\int_0^T \langle \chi_n, v \rangle \, dt &= \lim_{k\to\infty} \int_0^T \langle B_n u_k, v \rangle \, dt \\
&= \lim_{k\to\infty} \Big(\sum_{i=1}^N \int_{Q_T} a_i(x,t,T_n(u_k),\nabla u_k) D^i v \, dx \, dt + \int_{Q_T} g_n(x,t,u_k) v \, dx \, dt \\
&\quad + \frac{1}{n} \int_{Q_T} |u_k|^{p_0-2} u_k v \, dx \, dt \Big) \\
&= \sum_{i=1}^N \int_{Q_T} \varphi_i D^i v \, dx \, dt + \int_{Q_T} g_n(x,t,u) v \, dx \, dt + \frac{1}{n} \int_{Q_T} |u|^{p_0-2} u v \, dx \, dt.
\end{aligned} \tag{4.9}$$

Having in mind (4.5) and (4.9), we obtain

$$\begin{aligned}
\limsup_{k\to\infty} \langle B_n(u_k), u_k \rangle &= \limsup_{k\to\infty} \Big(\sum_{i=1}^N \int_{Q_T} a_i(x,t,T_n(u_k),\nabla u_k) D^i u_k \, dx \, dt \\
&\quad + \int_{Q_T} g_n(x,t,u_k) u_k \, dx \, dt + \frac{1}{n} \int_{Q_T} |u_k|^{p_0} \, dx \, dt \Big) \\
&\leqslant \sum_{i=1}^N \int_{Q_T} \varphi_i D^i u \, dx \, dt + \int_{Q_T} g_n(x,t,u) u \, dx \, dt + \frac{1}{n} \int_{Q_T} |u|^{p_0} \, dx \, dt.
\end{aligned} \tag{4.10}$$

Thanks to (4.7), we have

$$\int_{Q_T} g_n(x,t,u_k) u_k \, dx \, dt \longrightarrow \int_{Q_T} g_n(x,t,u) u \, dx \, dt. \tag{4.11}$$

Therefore

$$\begin{aligned}\limsup_{k\to\infty}&\left\{\sum_{i=1}^{N}\int_{Q_T}a_i(x,t,T_n(u_k),\nabla u_k)D^iu_k\,dx\,dt+\frac{1}{n}\int_{Q_T}|u_k|^{p_0}\,dx\,dt\right\}\\&\leqslant\sum_{i=1}^{N}\int_{Q_T}\varphi_iD^iu\,dx\,dt+\frac{1}{n}\int_{Q_T}|u|^{p_0}\,dx\,dt.\end{aligned}\tag{4.12}$$

On the other hand, using (3.4), we have

$$\begin{aligned}\sum_{i=1}^{N}\int_{Q_T}&(a_i(x,t,T_n(u_k),\nabla u_k)-a_i(x,t,T_n(u_k),\nabla u))(D^iu_k-D^iu)\,dx\,dt\\&+\frac{1}{n}\int_{Q_T}(|u_k|^{p_0-2}u_k-|u|^{p_0-2}u)(u_k-u)\,dx\,dt\geqslant 0.\end{aligned}\tag{4.13}$$

Then

$$\begin{aligned}&\sum_{i=1}^{N}\int_{Q_T}a_i(x,t,T_n(u_k),\nabla u_k)D^iu_k\,dx\,dt+\frac{1}{n}\int_{Q_T}|u_k|^{p_0}\,dx\,dt\\&\geqslant\sum_{i=1}^{N}\int_{Q_T}a_i(x,t,T_n(u_k),\nabla u)(D^iu_k-D^iu)\,dx\,dt+\sum_{i=1}^{N}\int_{Q_T}a_i(x,t,T_n(u_k),\nabla u_k)D^iu\,dx\,dt\\&\quad+\frac{1}{n}\int_{Q_T}|u|^{p_0-2}u(u_k-u)\,dx\,dt+\frac{1}{n}\int_{Q_T}|u_k|^{p_0-2}u_ku\,dx\,dt.\end{aligned}$$

Now, we have $T_n(u_k)\to T_n(u)$ in $L^{p_i}(Q_T)$, then $a_i(x,t,T_n(u_k),\nabla u)\to a_i(x,t,T_n(u),\nabla u)$ in $L^{p_i'}(Q_T)$. Thanks to (4.6) and (4.8), we obtain

$$\begin{aligned}\liminf_{k\to\infty}&\left\{\sum_{i=1}^{N}\int_{Q_T}a_i(x,t,T_n(u_k),\nabla u_k)D^iu_k\,dx\,dt+\frac{1}{n}\int_{Q_T}|u_k|^{p_0}\,dx\,dt\right\}\\&\geqslant\sum_{i=1}^{N}\int_{Q_T}\varphi_iD^iu\,dx\,dt+\frac{1}{n}\int_{Q_T}|u|^{p_0}\,dx\,dt,\end{aligned}$$

which implies, thanks to (4.12), that

$$\begin{aligned}\lim_{k\to\infty}&\left\{\sum_{i=1}^{N}\int_{Q_T}a_i(x,t,T_n(u_k),\nabla u_k)D^iu_k\,dx\,dt+\frac{1}{n}\int_{Q_T}|u_k|^{p_0}\,dx\,dt\right\}\\&=\sum_{i=1}^{N}\int_{Q_T}\varphi_iD^iu\,dx\,dt+\frac{1}{n}\int_{Q_T}|u|^{p_0}\,dx\,dt.\end{aligned}\tag{4.14}$$

By combining (4.9), (4.11) and (4.14), we deduce that

$$\int_0^T\langle B_nu_k,u_k\rangle\,dt\longrightarrow\int_0^T\langle\chi_n,u\rangle\,dt\quad\text{as }k\to\infty,$$

and in view of (4.14), we obtain

$$\begin{aligned}\sum_{i=1}^{N}\int_{Q_T}&(a_i(x,t,T_n(u_k),\nabla u_k)-a_i(x,t,T_n(u_k),\nabla u))(D^iu_k-D^iu)\,dx\,dt\\&+\frac{1}{n}\int_{Q_T}(|u_k|^{p_0-2}u_k-|u|^{p_0-2}u)(u_k-u)\,dx\,dt\longrightarrow 0\quad\text{as}\quad k\to+\infty.\end{aligned}$$

Hence, thanks to Lemma 3.2, we get

$$u_k \longrightarrow u \quad \text{in} \quad L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(Q_T)).$$

Thus, $D^i u_k \longrightarrow D^i u$ a.e. in Q_T. It follows that $a_i(x,t,T_n(u_k),\nabla u_k) \longrightarrow a_i(x,t,T_n(u),\nabla u)$ a.e. in Q_T, and in view of (3.2), we get

$$a_i(x,t,T_n(u_k),\nabla u_n) \rightharpoonup a_i(x,t,T_n(u),\nabla u) \quad \text{in} \quad L^{p_i'}(Q_T) \quad \text{for} \quad i=1,\dots,N.$$

Having in mind (4.7) and (4.8), we deduce that $\chi_n = B_n u$, which completes the proof of Lemma 4.1.

Consequently, in view of Lemma 4.1, there exists at least one weak solution $u_n \in L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$ of problem (4.2) (cf. [28], Theorem 2.7, page 180).

Step 2: A priori estimates. Taking $T_k(u_n)$ as a test function in (4.2), we obtain

$$\int_0^T \Big\langle \frac{\partial u_n}{\partial t}, T_k(u_n)\Big\rangle dt + \sum_{i=1}^N \int_{Q_T} a_i(x,t,T_n(u_n),\nabla u_n) D^i T_k(u_n)\,dx\,dt$$
$$+\int_{Q_T} g_n(x,t,u_n)T_k(u_n)\,dx\,dt + \frac{1}{n}\int_{Q_T} |u_n|^{p_0-2}u_n T_k(u_n)\,dx\,dt = \int_{Q_T} f_n T_k(u_n)\,dx\,dt. \tag{4.15}$$

Since $|\varphi_k(r)| \leqslant k|r|$, then

$$\begin{aligned}\int_0^T \Big\langle \frac{\partial u_n}{\partial t}, T_k(u_n)\Big\rangle dt &= \int_\Omega \int_0^T \frac{\partial u_n}{\partial t} T_k(u_n)\,dt\,dx = \int_\Omega\int_0^T \frac{\partial \varphi_k(u_n)}{\partial t}\,dt\,dx \\ &= \int_\Omega \varphi_k(u_n(T))\,dx - \int_\Omega \varphi_k(u_n(0))\,dx \\ &\geqslant \int_\Omega \varphi_k(u_n(T))\,dx - k\|u_0\|_{L^1(\Omega)}.\end{aligned} \tag{4.16}$$

In view of (3.6), the third and fourth terms on the right-hand side of (4.16) are positives, and thanks to (3.3), we get

$$\int_\Omega \varphi_k(u_n(T))\,dx + \alpha \sum_{i=1}^N \int_{Q_T} |D^i T_k(u_n)|^{p_i}\,dx\,dt \leqslant k(\|u_0\|_{L^1(\Omega)} + \|f\|_{L^1(Q_T)}). \tag{4.17}$$

On the other hand, we have

$$\begin{aligned}\sum_{i=1}^N \int_{Q_T} |D^i T_k(u_n)|^{p_i}\,dx\,dt &\geqslant \sum_{i=1}^N \int_{Q_T} |D^i T_k(u_n)|^{\underline{p}} - 1\,dx\,dt \\ &= \sum_{i=1}^N \|D^i T_k(u_n)\|_{L^{\underline{p}}(Q_T)}^{\underline{p}} - N\cdot \text{meas}(Q_T) \\ &\geqslant \frac{1}{N^{\underline{p}-1}} \|\nabla T_k(u_n)\|_{L^{\underline{p}}(Q_T)}^{\underline{p}} - N\cdot\text{meas}(Q_T).\end{aligned}$$

Since $\varphi_k(u_n(T)) \geqslant 0$, then there exists a constant C_1 that does not depend on n and k, such that

$$\|\nabla u_n\|_{L^{\underline{p}}(Q_T)} \leqslant C_1 k^{\frac{1}{\underline{p}}} \qquad \text{for} \quad k \geqslant 1. \tag{4.18}$$

Therefore, for any $k \geqslant 1$, we have

$$\begin{aligned} k \text{ meas}\{|u_n| > k\} &= \int_{\{|u_n|>k\}} |T_k(u_n)|\,dx\,dt \leqslant \int_{Q_T} |T_k(u_n)|\,dx\,dt \\ &\leqslant \|1\|_{L^{\underline{p}'}(Q_T)} \cdot \|T_k(u_n)\|_{L^{\underline{p}}(Q_T)} \\ &\leqslant C_2 \|\nabla T_k(u_n)\|_{L^{\underline{p}'}(Q_T)} \\ &\leqslant C_3 k^{\frac{1}{\underline{p}}}, \end{aligned}$$

which implies that

$$\text{meas}\{|u_n| > k\} \leqslant C_3 \frac{1}{k^{1-\frac{1}{\underline{p}}}} \longrightarrow 0 \quad \text{as} \quad k \to +\infty. \tag{4.19}$$

We have for any $\delta > 0$

$$\begin{aligned} \text{meas}\{|u_n - u_m| > \delta\} \leqslant\ & \text{meas}\{|u_n| > k\} + \text{meas}\{|u_m| > k\} \\ & + \text{meas}\{|T_k(u_n) - T_k(u_m)| > \delta\}. \end{aligned} \tag{4.20}$$

Using (4.19), we get that for all $\varepsilon > 0$, there exists $k_0 > 0$ such that

$$\text{meas}\{|u_n| > k\} \leqslant \frac{\varepsilon}{3} \qquad \text{and} \qquad \text{meas}\{|u_m| > k\} \leqslant \frac{\varepsilon}{3} \qquad \forall k \geqslant k_0(\varepsilon). \tag{4.21}$$

On the other hand, thanks to (4.17), we have $(T_k(u_n))_n$ is bounded in $L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$. Then there exists a subsequence still denoted by $(T_k(u_n))_n$ and a measurable function $\eta_k \in L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$ such that

$$T_k(u_n) \rightharpoonup \eta_k \quad \text{in} \quad L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega)) \quad \text{as} \quad n \to +\infty.$$

Now the compact embedding (2.8) gives

$$T_k(u_n) \longrightarrow \eta_k \quad \text{in} \quad L^1(Q_T) \quad \text{and a.e in} \quad Q_T.$$

Thus, we can assume that $(T_k(u_n))_n$ is a Cauchy sequence in measure in Q_T, then for all $k > 0$ and $\delta, \varepsilon > 0$, there exists $n_0 = n_0(k, \delta, \varepsilon)$ such that

$$\text{meas}\{|T_k(u_n) - T_k(u_m)| > \delta\} \leqslant \frac{\varepsilon}{3} \qquad \forall n, m \geqslant n_0. \tag{4.22}$$

By combining (4.20)–(4.22), we deduce that for all ε, $\delta > 0$, there exists $n_0 = n_0(\delta, \varepsilon)$ such that

$$\text{meas}\{|u_n - u_m| > \delta\} \leqslant \varepsilon \qquad \forall n, m \geqslant n_0. \tag{4.23}$$

If follows that $(u_n)_n$ is a Cauchy sequence in measure. Hence, there exists a subsequence still denoted by $(u_n)_n$, such that

$$u_n \longrightarrow u \qquad \text{a.e in} \quad Q_T.$$

We deduce that

$$T_k(u_n) \rightharpoonup T_k(u) \qquad \text{in} \quad L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega)), \tag{4.24}$$

and in view of Lebesgue's dominated convergence theorem, we get

$$T_k(u_n) \longrightarrow T_k(u) \qquad \text{in} \quad L^{p_0}(Q_T). \tag{4.25}$$

Step 3: The equi-integrability of the sequences $(g_n(x,t,u_n))_n$ *and* $\left(\frac{1}{n}|u_n|^{p_0-2}u_n\right)_n$. In this step, we show that

$$g_n(x,t,u_n) \to g(x,t,u) \quad \text{and} \quad \frac{1}{n}|u_n|^{p_0-2}u_n \to 0 \quad \text{strongly in} \quad L^1(Q_T).$$

Indeed, we have $g_n(x,t,u_n) \to g(x,t,u)$ and $\frac{1}{n}|u_n|^{p_0-2}u_n \to 0$ a.e. Q_T. By using Vitali's theorem, it suffices to prove that $(g_n(x,u_n,\nabla u_n))_n$ and $(\frac{1}{n}|u_n|^{p_0-2}u_n)_n$ are uniformly equi-integrable in Q_T.

Taking $T_{h+1}(u_n) - T_h(u_n)$ as a test function in (4.2), we obtain

$$\begin{aligned}
&\int_0^T \Big\langle \frac{\partial u_n}{\partial t}, T_{h+1}(u_n) - T_h(u_n) \Big\rangle \, dt \\
&\quad + \sum_{i=1}^N \int_{Q_T} a_i(x,t,T_n(u_n),\nabla u_n) \cdot (D^i T_{h+1}(u_n) - D^i T_h(u_n)) \, dx\, dt \\
&\quad + \int_{Q_T} g_n(x,t,u_n)(T_{h+1}(u_n) - T_h(u_n)) \, dx\, dt \\
&\quad + \frac{1}{n} \int_{Q_T} |u_n|^{p_0-2} u_n (T_{h+1}(u_n) - T_h(u_n)) \, dx\, dt \\
&= \int_{Q_T} f_n \cdot (T_{h+1}(u_n) - T_h(u_n)) \, dx\, dt.
\end{aligned}$$

On the one hand, we have

$$\begin{aligned}
\int_0^T \Big\langle \frac{\partial u_n}{\partial t}, T_{h+1}(u_n) - T_h(u_n) \Big\rangle \, dt &= \int_\Omega \int_0^T \frac{\partial \varphi_{h+1}(u_n)}{\partial t} \, dt\, dx - \int_\Omega \int_0^T \frac{\partial \varphi_h(u_n)}{\partial t} \, dt\, dx \\
&= \int_\Omega \varphi_{h+1}(u_n(T)) - \varphi_{h+1}(u_{0,n}) \, dx \\
&\quad - \int_\Omega \varphi_h(u_n(T)) - \varphi_h(u_{0,n}) \, dx.
\end{aligned}$$

Since

$$\begin{aligned}
\int_\Omega \varphi_{h+1}(u_n(T)) \, dx - \int_\Omega \varphi_h(u_n(T)) \, dx &= \int_{\{h \leqslant |u_n(T)| < h+1\}} \Big(\frac{u_n^2(T)}{2} - h|u_n(T)| + \frac{h^2}{2} \Big) \, dx \\
&\quad + \int_{\{h+1 \leqslant |u_n(T)|\}} \Big(|u_n(T)| - h - \frac{1}{2} \Big) \, dx \geqslant 0,
\end{aligned} \tag{4.26}$$

then thanks to (3.3) and (3.6), we deduce that

$$\begin{aligned}
&\alpha\sum_{i=1}^{N}\int_0^T\int_{\{h\leqslant|u_n|<h+1\}}|D^iu_n|^{p_i}\,dx\,dt+\int_0^T\int_{\{h+1\leqslant|u_n|\}}|g_n(x,t,u_n)|\,dx\,dt\\
&\quad+\frac{1}{n}\int_0^T\int_{\{h+1\leqslant|u_n|\}}|u_n|^{p_0-1}\,dx\,dt\\
&\leqslant\sum_{i=1}^{N}\int_0^T\int_{\{h\leqslant|u_n|<h+1\}}a_i(x,t,T_n(u_n),\nabla u_n)\cdot D^iu_n\,dx\,dt\\
&\quad+\int_0^T\int_{\{h\leqslant|u_n|\}}g_n(x,t,u_n)(T_{h+1}(u_n)-T_h(u_n))\,dx\,dt\\
&\quad+\frac{1}{n}\int_0^T\int_{\{h\leqslant|u_n|\}}|u_n|^{p_0-2}u_n(T_{h+1}(u_n)-T_h(u_n))\,dx\,dt\\
&\leqslant\int_{Q_T}f_n\cdot(T_{h+1}(u_n)-T_h(u_n))\,dx\,dt+\int_\Omega\varphi_{h+1}(u_{0,n})\,dx-\int_\Omega\varphi_h(u_{0,n})\,dx.
\end{aligned}$$

The terms on the right-hand side of the inequality above can be interpreted as follows

$$\Big|\int_{Q_T}f_n\cdot(T_{h+1}(u_n)-T_h(u_n))\,dx\,dt\Big|\leqslant\int_{\{|u_n|\geqslant h\}}|f|\,dx\,dt\longrightarrow 0\qquad\text{as}\quad h\to\infty,$$

and since $u_0\in L^1(\Omega)$, then

$$\begin{aligned}
&\int_\Omega\varphi_{h+1}(u_{0,n})\,dx-\int_\Omega\varphi_h(u_{0,n})\,dx\\
&=\int_{\{h\leqslant|u_{0,n}|<h+1\}}\Big(\frac{|u_{0,n}|^2}{2}-h|u_{0,n}|+\frac{h^2}{2}\Big)\,dx+\int_{\{h+1\leqslant|u_{0,n}|\}}\Big(|u_{0,n}|-h-\frac{1}{2}\Big)\,dx\\
&\leqslant\int_{\{h\leqslant|u_{0,n}|<h+1\}}\frac{1}{2}\,dx+\int_{\{h+1\leqslant|u_{0,n}|\}}\Big(|u_0|-h-\frac{1}{2}\Big)\,dx\longrightarrow 0\quad\text{as}\quad h\to\infty.
\end{aligned}$$

Therefore, we conclude that

$$\sum_{i=1}^{N}\int_0^T\int_{\{h\leqslant|u_n|<h+1\}}a_i(x,t,T_n(u_n),\nabla u_n)\cdot D^iu_n\,dx\,dt\longrightarrow 0\quad\text{as}\quad h\to\infty,\tag{4.27}$$

$$\int_0^T\int_{\{h+1\leqslant|u_n|\}}|g_n(x,t,u_n)|\,dx\,dt\longrightarrow 0\quad\text{as}\quad h\to\infty,\tag{4.28}$$

and

$$\frac{1}{n}\int_0^T\int_{\{h+1\leqslant|u_n|\}}|u_n|^{p_0-1}\,dx\,dt\longrightarrow 0\quad\text{as}\quad h\to\infty.\tag{4.29}$$

Moreover, thanks to (4.28)–(4.29), we have $\forall\eta>0$, $\exists h(\eta)>0$ such that

$$\int_0^T\int_{\{h(\eta)\leqslant|u_n|\}}|g_n(x,t,u_n)|\,dx\,dt+\frac{1}{n}\int_0^T\int_{\{h(\eta)\leqslant|u_n|\}}|u_n|^{p_0-1}\,dx\,dt\leqslant\frac{\eta}{2}.\tag{4.30}$$

On the other hand, for any measurable subset $E \subset Q_T$, we have

$$\begin{aligned} &\int_E |g_n(x,t,u_n)|\,dx\,dt + \frac{1}{n}\int_E |u_n|^{p_0-1}\,dx\,dt \\ &\leqslant \int_E |g_n(x,t,T_{h(\eta)}(u_n))|\,dx\,dt + \frac{1}{n}\int_E |T_{h(\eta)}(u_n)|^{p_0-1}\,dx\,dt \\ &\quad + \int_{\{h(\eta)\leqslant|u_n|\}} |g_n(x,t,u_n)|\,dx\,dt + \frac{1}{n}\int_{\{h(\eta)\leqslant|u_n|\}} |u_n|^{p_0-1}\,dx\,dt, \end{aligned} \tag{4.31}$$

and there exists $\lambda(\eta) > 0$ such that

$$\int_E |g_n(x,t,T_{h(\eta)}(u_n))|\,dx\,dt + \frac{1}{n}\int_E |T_{h(\eta)}(u_n)|^{p_0-1}\,dx\,dt \leqslant \frac{\eta}{2} \quad \text{for meas}(E) \leqslant \lambda(\eta). \tag{4.32}$$

Finally, by combining (4.30), (4.31) and (4.32), we obtain

$$\int_E |g_n(x,t,u_n)|\,dx\,dt + \frac{1}{n}\int_E |u_n|^{p_0-1}\,dx\,dt \leqslant \eta, \quad \text{with meas}(E) \leqslant \lambda(\eta). \tag{4.33}$$

Thus $(g_n(x,t,u_n))_n$ and $(|u_n|^{p_0-1})_n$ are uniformly equi-integrable, and in view of Vitali's Theorem we deduce that

$$g_n(x,t,u_n) \longrightarrow g(x,t,u) \quad \text{and} \quad \frac{1}{n}|u_n|^{p_0-1} \longrightarrow 0 \quad \text{in} \quad L^1(Q_T). \tag{4.34}$$

Step 4: The weak convergence of $(S_h(u_n))_t$ *in* $L^{\vec{p}'}(0,T;W^{-1,\vec{p}'}(\Omega)) + L^1(Q_T)$. Let $S_h(\cdot)$ be an increasing function $C^2(\mathbb{R})$, such that $S_h(r) = r$ for $|r| \leqslant h$ and $\text{supp}(S_h') \subset [-h-1, h+1]$, then $\text{supp}(S_h'') \subset [-h-1,-h] \cup [h,h+1]$.

Let $v \in L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega)) \cap L^\infty(Q_T)$. By taking $S_h'(u_n)v$ as a test function in (4.2), we get

$$\begin{aligned} &\int_0^T \Big\langle \frac{\partial u_n}{\partial t}, S_h'(u_n)v \Big\rangle\,dt + \sum_{i=1}^N \int_{Q_T} a_i(x,t,T_n(u_n),\nabla u_n)(S_h'(u_n)D^i v + S_h''(u_n) v D^i u_n)\,dx\,dt \\ &\quad + \int_{Q_T} g_n(x,t,u_n) S_h'(u_n) v\,dx\,dt + \frac{1}{n}\int_{Q_T} |u_n|^{p_0-2} u_n S_h'(u_n) v\,dx\,dt = \int_{Q_T} f_n S_h'(u_n) v\,dx\,dt. \end{aligned}$$

Then

$$\begin{aligned} \Big|\int_0^T \Big\langle \frac{\partial S_h(u_n)}{\partial t}, v \Big\rangle\,dt\Big| &\leqslant \sum_{i=1}^N \int_{Q_T} |a_i(x,t,T_n(u_n),\nabla u_n)|\,|S_h'(u_n)D^i v + S_h''(u_n) v D^i u_n|\,dx\,dt \\ &\quad + \int_{Q_T} |g_n(x,t,u_n)|\,|S_h'(u_n)v|\,dx\,dt \\ &\quad + \frac{1}{n}\int_{Q_T} |u_n|^{p_0-1}\,|S_h'(u_n)v|\,dx\,dt \\ &\quad + \int_{Q_T} |f_n|\,|S_h'(u_n)v|\,dx\,dt. \end{aligned} \tag{4.35}$$

For the first term on the right-hand side of (4.35), we have

$$
\begin{aligned}
&\int_{Q_T} |a_i(x,t,T_n(u_n),\nabla u_n)|\,|S_h'(u_n)D^i v + S_h''(u_n)vD^i u_n|\,dx\,dt \\
&\leqslant \int_0^T \int_{\{|u_n|\leqslant h+1\}} \beta(K_i(x,t)+|T_n(u_n)|^{p_i-1}+|D^i u_n|^{p_i-1}) \\
&\quad \times(|S_h'(u_n)||D^i v| + |S_h''(u_n)||v||D^i u_n|)\,dx\,dt \\
&\leqslant 2\beta\left(\|K_i(x,t)\|_{L^{p_i'}(Q_T)} + \|T_{h+1}(u_n)\|^{p_i-1}_{L^{p_i}(Q_T)} + \|D^i T_{h+1}(u_n)\|^{p_i-1}_{L^{p_i}(Q_T)}\right) \\
&\quad \times\left(\|S_h'(\cdot)\|_{L^\infty(\mathbb{R})}\|D^i v\|_{L^{p_i}(Q_T)} + \|S_h''(\cdot)\|_{L^\infty(\mathbb{R})}\|v\|_{L^\infty(Q_T)}\|D^i T_{h+1}(u_n)\|_{L^{p_i}(Q_T)}\right) \\
&\leqslant C_4(\|v\|_{L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))} + \|v\|_{L^\infty(Q_T)}).
\end{aligned}
\tag{4.36}
$$

Concerning the last three terms on the right-hand side of (4.35), we get

$$
\begin{aligned}
&\int_{Q_T} |g_n(x,t,u_n)|\,|S_h'(u_n)v|\,dx\,dt \\
&\quad + \frac{1}{n}\int_{Q_T} |u_n|^{p_0-1}\,|S_h'(u_n)v|\,dx\,dt + \int_{Q_T} |f_n|\,|S_h'(u_n)v|\,dx\,dt \\
&\leqslant \int_{Q_T}\left(b(x,t)+(\gamma+\frac{1}{n})|u_n|^{p_0-1}\right)|S_h'(u_n)v|\,dx\,dt + \int_{Q_T}|f_n|\,|S_h'(u_n)v|\,dx\,dt \\
&\leqslant 2\left(\|b(x,t)\|_{L^{p_0'}(Q_T)} + 2\|T_{h+1}(u_n)\|^{p_0-1}_{L^{p_0}(Q_T)}\right)\|S_h'(\cdot)\|_{L^\infty(\mathbb{R})}\|v\|_{L^{p_0}(Q_T)} \\
&\quad + \|f\|_{L^1(Q_T)}\cdot\|S_h'(\cdot)\|_{L^\infty(\mathbb{R})}\cdot\|v\|_{L^\infty(Q_T)} \\
&\leqslant C_5(\|v\|_{L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))} + \|v\|_{L^\infty(Q_T)}).
\end{aligned}
\tag{4.37}
$$

Using (4.35)–(4.37), we obtain

$$
\left|\int_0^T \left\langle \frac{\partial S_h(u_n)}{\partial t}, v\right\rangle dt\right| \leqslant C_6(\|v\|_{L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))} + \|v\|_{L^\infty(Q_T)}) \tag{4.38}
$$

for $v\in L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))\cap L^\infty(Q_T)$, with C_6 is a constant that does not depend on n. Hence $\left(\frac{\partial S_h(u_n)}{\partial t}\right)_n$ is bounded in $L^{\vec{p}'}(0,T;W^{-1,\vec{p}'}(\Omega))+L^1(Q_T)$ and

$$
\frac{\partial S_h(u_n)}{\partial t} \rightharpoonup \frac{\partial S_h(u)}{\partial t} \quad \text{in} \quad L^{\vec{p}'}(0,T;W^{-1,\vec{p}'}(\Omega))+L^1(Q_T). \tag{4.39}
$$

Step 5: Convergence of the gradient. Let $k\leqslant h$ and n large enough. By taking $S_h'(u_n)(T_k(u_n)-T_k(u))$ as a test function in (4.2), we obtain

$$
\mathscr{I}^1_{n,h} + \mathscr{I}^2_{n,h} + \mathscr{I}^3_{n,h} + \mathscr{I}^4_{n,h} = \mathscr{I}^5_{n,h}, \tag{4.40}
$$

where

$$\begin{aligned}
\mathscr{I}^1_{n,h} &= \int_0^T \int_\Omega \frac{\partial S_h(u_n)}{\partial t}(T_k(u_n)-T_k(u))\,dx\,dt\\
\mathscr{I}^2_{n,h} &= \sum_{i=1}^N \int_{Q_T} S'_h(u_n)a_i(x,t,T_n(u_n),\nabla u_n)(D^iT_k(u_n)-D^iT_k(u))\,dx\,dt\\
\mathscr{I}^3_{n,h} &= \sum_{i=1}^N \int_{Q_T} (T_k(u_n)-T_k(u))S''_h(u_n)a_i(x,t,T_n(u_n),\nabla u_n)D^iu_n\,dx\,dt\\
\mathscr{I}^4_{n,h} &= \int_{Q_T} g_n(x,t,u_n)S'_h(u_n)(T_k(u_n)-T_k(u))\,dx\,dt\\
&\quad+\frac{1}{n}\int_{Q_T} |u_n|^{p_0-2}u_nS'_h(u_n)(T_k(u_n)-T_k(u))\,dx\,dt\\
\mathscr{I}^5_{n,h} &= \int_{Q_T} f_nS'_h(u_n)(T_k(u_n)-T_k(u))\,dx\,dt.
\end{aligned} \tag{4.41}$$

The first term. In view of (4.39), we have $\dfrac{\partial S_h(u_n)}{\partial t} \rightharpoonup \dfrac{\partial S_h(u)}{\partial t}$ in $L^{\vec{p}'}(0,T;W^{-1,\vec{p}'}(\Omega)) + L^1(Q_T)$, then

$$\begin{aligned}
\liminf_{n\to\infty} \mathscr{I}^1_{n,h} &= \liminf_{n\to\infty}\int_{Q_T} \frac{\partial S_h(u_n)}{\partial t}T_k(u_n)\,dx\,dt - \lim_{n\to\infty}\int_{Q_T}\frac{\partial S_h(u_n)}{\partial t}T_k(u)\,dx\,dt\\
&= \liminf_{n\to\infty}\int_{Q_T} \frac{\partial S_h(u_n)}{\partial t}T_k(S_h(u_n))\,dx\,dt - \int_{Q_T}\frac{\partial S_h(u)}{\partial t}T_k(S_h(u))\,dx\,dt\\
&= \liminf_{n\to\infty}\int_\Omega \varphi_k(S_h(u_n(T)))-\varphi_k(S_h(u_{0,n}))\,dx\\
&\quad -\int_\Omega \varphi_k(S_h(u(T)))-\varphi_k(S_h(u_0))\,dx\\
&= \liminf_{n\to\infty}\int_\Omega \varphi_k(S_h(u_n(T)))\,dx - \int_\Omega \varphi_k(S_h(u(T)))\,dx.
\end{aligned} \tag{4.42}$$

Now, we show that the right-hand side of (4.42) is positive. By using $S'_h(u_n)T_k(u_n)$ as a test function in (4.2), we get

$$\begin{aligned}
&\int_0^T \left\langle \frac{\partial S_h(u_n)}{\partial t}, T_k(u_n)\right\rangle dt + \int_{Q_T} S''_h(u_n)T_k(u_n)\, a(x,t,T_n(u_n),\nabla u_n)\cdot\nabla u_n\,dx\,dt\\
&\quad+\int_{Q_T} S'_h(u_n)\, a(x,t,T_n(u_n),\nabla u_n)\cdot\nabla T_k(u_n)\,dx\,dt\\
&\quad+\int_{Q_T} g_n(x,t,u_n)S'_h(u_n)T_k(u_n)\,dx\,dt + \int_{Q_T}|u_n|^{p_0-2}u_nS'_h(u_n)T_k(u_n)\,dx\,dt\\
&\quad=\int_{Q_T} f_nS'_h(u_n)T_k(u_n)\,dx\,dt.
\end{aligned} \tag{4.43}$$

Since $S_h'(u_n) \geqslant 0$, then the last three terms on the left-hand side of (4.43) are nonnegative. Thus

$$\begin{aligned}
&\int_\Omega \varphi_k(S_h(u_n(T)))\,dx - \int_\Omega \varphi_k(S_h(u_{0,n}))\,dx \\
&\leqslant k\|S_h''(\cdot)\|_{L^\infty(\mathbb{R})} \int_{\{h\leqslant |u_n|<h+1\}} |a(x,t,T_{h+1}(u_n),\nabla T_{h+1}(u_n))|\,|\nabla T_{h+1}(u_n)|\,dx\,dt \\
&\qquad + k\|S_h'(\cdot)\|_{L^\infty(\mathbb{R})}\|f\|_1 \\
&\leqslant kC_7,
\end{aligned}$$

with C_7 is a constant that does not depend on n. Then

$$\int_\Omega \varphi_k(S_h(u_n(T)))\,dx \leqslant kC_7 + \int_\Omega \varphi_k(S_h(u_0))\,dx.$$

Now, we have $\varphi_k(S_h(u_n(T))) \geqslant 0$ and $\varphi_k(S_h(u_n(T))) \to \varphi_k(S_h(u(T)))$ a.e. in Ω, and thanks to Fatou's lemma, we obtain

$$\int_\Omega \varphi_k(S_h(u(T)))\,dx \leqslant \liminf_{n\to\infty} \int_\Omega \varphi_k(S_h(u_n(T)))\,dx.$$

Hence, thanks to (4.42), we deduce that

$$\liminf_{n\to\infty} \mathscr{I}_{n,h}^1 = \liminf_{n\to\infty} \int_\Omega \varphi_k(S_h(u_n(T)))\,dx - \int_\Omega \varphi_k(S_h(u(T)))\,dx \geqslant 0. \tag{4.44}$$

The second term. We have $S_h'(s) \geqslant 0$ and $S_h'(s) = 1$ for $|s| \leqslant k$, with $\operatorname{supp}(S_h') \subset [-h-1, h+1]$, then

$$\begin{aligned}
\mathscr{I}_{n,h}^2 &= \sum_{i=1}^N \int_0^T \int_{\{|u_n|\leqslant k\}} a_i(x,t,T_k(u_n),\nabla T_k(u_n))(D^iT_k(u_n) - D^iT_k(u))\,dx\,dt \\
&\quad - \sum_{i=1}^N \int_0^T \int_{\{k<|u_n|\leqslant h+1\}} S_h'(u_n)a_i(x,t,T_{h+1}(u_n),\nabla T_{h+1}(u_n))D^iT_k(u)\,dx\,dt \\
&\geqslant \sum_{i=1}^N \int_{Q_T} \Big(a_i(x,t,T_k(u_n),\nabla T_k(u_n)) - a_i(x,t,T_k(u_n),\nabla T_k(u))\Big) \\
&\quad \times (D^iT_k(u_n) - D^iT_k(u))\,dx\,dt \\
&\quad + \sum_{i=1}^N \int_{Q_T} a_i(x,t,T_k(u_n),\nabla T_k(u))(D^iT_k(u_n) - D^iT_k(u))\,dx\,dt \\
&\quad + \sum_{i=1}^N \int_{\{|u_n|>k\}} a_i(x,t,T_k(u_n),\nabla T_k(u_n))D^iT_k(u)\,dx\,dt \\
&\quad - \sum_{i=1}^N \|S_h'(\cdot)\|_{L^\infty(\mathbb{R})} \int_{\{k<|u_n|\leqslant h+1\}} |a_i(x,t,T_{h+1}(u_n),\nabla T_{h+1}(u_n))||D^iT_k(u)|\,dx\,dt.
\end{aligned} \tag{4.45}$$

In view of Lebesgue's dominated convergence theorem, we have $T_k(u_n) \to T_k(u)$ in $L^{p_i}(Q_T)$. Thus, $a_i(x,t,T_k(u_n),\nabla T_k(u)) \to a_i(x,t,T_k(u),\nabla T_k(u))$ in $L^{p'_i}(Q_T)$ and since $D^i T_k(u_n) \rightharpoonup D^i T_k(u)$ in $L^{p_i}(Q_T)$, then

$$\int_{Q_T} a_i(x,t,T_k(u_n),\nabla T_k(u))(D^i T_k(u_n) - D^i T_k(u))\,dx\,dt \longrightarrow 0 \qquad \text{as} \quad n\to\infty. \tag{4.46}$$

Using (3.3) and the continuity of the Carathéodory function, we have $a_i(x,t,s,0) = 0$. Hence

$$\begin{aligned}\int_{\{|u_n|>k\}} a_i(x,t,T_k(u_n),\nabla T_k(u_n)) D^i T_k(u)\,dx\,dt &= \int_{\{|u_n|>k\}} a_i(x,t,T_k(u_n),0) D^i T_k(u)\,dx\,dt \\ &= 0. \end{aligned} \tag{4.47}$$

For the last term of the right-hand side of (4.45), we have $(a_i(x,t,T_{h+1}(u_n),\nabla T_{h+1}(u_n)))_n$ is bounded in $L^{p'_i}(Q_T)$, then there exists $\xi_i \in L^{p'_i}(Q_T)$ such that $a_i(x,t,T_{h+1}(u_n),\nabla T_{h+1}(u_n)) \rightharpoonup \xi_i$ in $L^{p'_i}(Q_T)$. It follows that

$$\begin{aligned}&\int_{\{k<|u_n|\leqslant h+1\}} |a_i(x,t,T_{h+1}(u_n),\nabla T_{h+1}(u_n))||D^i T_k(u)|\,dx\,dt \\ &\qquad \longrightarrow \int_{\{k<|u|\leqslant h+1\}} \xi_i\, |D^i T_k(u)|\,dx\,dt = 0. \end{aligned} \tag{4.48}$$

By combining (4.46)–(4.48), we deduce that

$$\begin{aligned}\mathscr{I}^2_{n,h} \geqslant \sum_{i=1}^{N} \int_{Q_T} &\big(a_i(x,t,T_k(u_n),\nabla T_k(u_n)) - a_i(x,t,T_k(u_n),\nabla T_k(u))\big) \\ &\times (D^i T_k(u_n) - D^i T_k(u))dxdt + \varepsilon_2(n). \end{aligned} \tag{4.49}$$

The third term. We have $\mathrm{supp}(S''_h) \subset [-h-1,-h]\cup[h,h+1]$. Moreover, in view of Young's inequality, we get

$$\begin{aligned}\mathscr{I}^3_{n,h} &\leqslant \|S''_h(\cdot)\|_{L^\infty(\mathbb{R})} \sum_{i=1}^{N} \int_0^T \int_{\{h<|u_n|\leqslant h+1\}} |T_k(u_n) - T_k(u)|\;|D^i T_{h+1}(u_n)| \\ &\quad \times |a_i(x,t,T_{h+1}(u_n),\nabla T_{h+1}(u_n))|\,dx\,dt \\ &\leqslant \beta \|S''_h(\cdot)\|_{L^\infty(\mathbb{R})} \sum_{i=1}^{N} \int_0^T \int_{\{h<|u_n|\leqslant h+1\}} |T_k(u_n) - T_k(u)|\;|D^i T_{h+1}(u_n)| \\ &\quad \times (K_i(x,t) + |T_{h+1}(u_n)|^{p_i-1} + |D^i T_{h+1}(u_n)|^{p_i-1})\,dx\,dt \\ &\leqslant \beta \|S''_h(\cdot)\|_{L^\infty(\mathbb{R})} \sum_{i=1}^{N} \int_0^T \int_{\{h<|u_n|\leqslant h+1\}} |T_k(u_n) - T_k(u)| \frac{|K_i(x,t)|^{p'_i}}{p'_i}\,dx\,dt \\ &\quad + \beta \|S''_h(\cdot)\|_{L^\infty(\mathbb{R})} \sum_{i=1}^{N} \int_0^T \int_{\{h<|u_n|\leqslant h+1\}} |T_k(u_n) - T_k(u)| \frac{|T_{h+1}(u_n)|^{p_i}}{p'_i}\,dx\,dt \\ &\quad + 2k\beta \|S''_h(\cdot)\|_{L^\infty(\mathbb{R})} (\frac{2}{p_i} + 1) \sum_{i=1}^{N} \int_0^T \int_{\{h<|u_n|\leqslant h+1\}} |D^i T_{h+1}(u_n)|^{p_i}\,dx\,dt. \end{aligned}$$

Since $T_k(u_n)-T_k(u)\rightharpoonup 0$ weak-$\star$ in $L^\infty(Q_T)$, then

$$\int_{\{h<|u_n|\leqslant h+1\}}|T_k(u_n)-T_k(u)|\frac{|K_i(x,t)|^{p_i'}}{p_i'}\,dx\,dt\longrightarrow 0 \qquad \text{as} \quad n\to\infty,$$

and

$$\int_{\{h<|u_n|\leqslant h+1\}}|T_k(u_n)-T_k(u)|\frac{|T_{h+1}(u_n)|^{p_i}}{p_i'}\,dx\,dt\longrightarrow 0 \qquad \text{as} \quad n\to\infty.$$

Thanks to (4.27), we have

$$\int_{\{h<|u_n|\leqslant h+1\}}|D^iT_{h+1}(u_n)|^{p_i}\,dx\,dt\longrightarrow 0 \qquad \text{as} \quad h\to\infty.$$

Hence, it follows that

$$\mathscr{I}^3_{n,h}\longrightarrow 0 \qquad \text{as} \quad n \text{ and } h\to\infty. \tag{4.50}$$

The fourth and last terms. We have $T_k(u_n)-T_k(u)\rightharpoonup 0$ weak-$\star$ in $L^\infty(Q_T)$, and thanks to (4.34), we obtain

$$\begin{aligned}|\mathscr{I}^4_{n,h}|\leqslant \|S_h'(\cdot)\|_{L^\infty(\mathbb{R})}\int_{Q_T}|g_n(x,t,u_n)|\,|T_k(u_n)-T_k(u)|\,dx\,dt\\ +\frac{\|S_h'(\cdot)\|_{L^\infty(\mathbb{R})}}{n}\int_{Q_T}|T_k(u_n)|^{p_0-1}|T_k(u_n)-T_k(u)|\,dx\,dt\longrightarrow 0 \quad \text{as} \quad n\to\infty.\end{aligned} \tag{4.51}$$

Also, we have $f_n\to f$ in $L^1(Q_T)$, then

$$\mathscr{I}^5_{n,h}\leqslant \|S_h'(\cdot)\|_{L^\infty(\mathbb{R})}\int_{Q_T}|f_n|\,|T_k(u_n)-T_k(u)|\,dx\,dt\longrightarrow 0 \qquad \text{as} \quad n\to 0. \tag{4.52}$$

Combining (4.40), (4.44) and (4.49)–(4.52), we deduce that

$$\begin{aligned}\sum_{i=1}^N\int_{Q_T}\Big(a_i(x,t,T_k(u_n),\nabla T_k(u_n))-a_i(x,t,T_k(u_n),\nabla T_k(u))\Big)\\ \times(D^iT_k(u_n)-D^iT_k(u))\,dx\,dt\leqslant\varepsilon_5(n,h).\end{aligned}$$

By letting n and h tend to infinity, we conclude that

$$\begin{aligned}\lim_{n\to\infty}\Big(\sum_{i=1}^N\int_{Q_T}\Big(a_i(x,t,T_k(u_n),\nabla T_k(u_n))-a_i(x,t,T_k(u_n),\nabla T_k(u))\Big)\\ \times(D^iT_k(u_n)-D^iT_k(u))\,dx\,dt\\ +\int_{Q_T}(|T_k(u_n)|^{p_0-2}T_k(u_n)-|T_k(u)|^{p_0-2}T_k(u))(T_k(u_n)-T_k(u))\,dx\,dt\Big)=0.\end{aligned} \tag{4.53}$$

Now, in view of Lemma 3.2, then

$$T_k(u_n) \longrightarrow T_k(u) \quad \text{in} \quad L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega)) \qquad \forall k>0. \tag{4.54}$$

Therefore $\nabla u_n \longrightarrow \nabla u$ a.e in Q_T.

Step 6: The convergence of u_n in $C([0,T];L^1(\Omega))$. Let u_n (resp. u_m) be the weak solution of the approximate problem (4.2) for the integer n (resp. m). For $0<s\leqslant T$, taking $T_1(u_n-u_m)\cdot\chi_{[0,s]}$ as a test function allows us to obtain

$$\begin{aligned}
&\int_\Omega\int_0^s \frac{\partial\varphi_1(u_n-u_m)}{\partial t}\,dt\,dx\\
&\quad+\sum_{i=1}^N\int_0^s\int_\Omega\Big(a_i(x,t,T_n(u_n),\nabla u_n)-a_i(x,t,T_m(u_m),\nabla u_m)\Big)D^iT_1(u_n-u_m)\,dx\,dt\\
&\quad+\int_0^s\int_\Omega\Big(g_n(x,t,u_n)-g_m(x,t,u_m)\Big)T_1(u_n-u_m)\,dx\,dt\\
&\quad+\int_0^s\int_\Omega\Big(\frac{1}{n}|u_n|^{p_0-2}u_n-\frac{1}{m}|u_m|^{p_0-2}u_m\Big)T_1(u_n-u_m)\,dx\,dt\\
&=\int_0^s\int_\Omega(f_n-f_m)\cdot T_1(u_n-u_m)\,dx\,dt.
\end{aligned} \tag{4.55}$$

On one hand, we have

$$\begin{aligned}
\int_\Omega\int_0^s \frac{\partial\varphi_1(u_n-u_m)}{\partial t}\,dt\,dx &= \int_\Omega \varphi_1(u_n(s)-u_m(s))-\varphi_1(u_n(0)-u_m(0))\,dx\\
&=\int_\Omega\varphi_1(u_n(s)-u_m(s))\,dx-\int_\Omega\varphi_1(u_{0,n}-u_{0,m})\,dx.
\end{aligned}$$

In what concern the second term on the left-hand side of (4.55), since $\nabla T_1(u_n-u_m)=(\nabla u_n-\nabla u_m)\cdot\chi_{\{|u_n-u_m|\leqslant 1\}}$, we deduce that (see Appendix)

$$\int_0^s\int_\Omega(a_i(x,t,T_n(u_n),\nabla u_n)-a_i(x,t,T_m(u_m),\nabla u_m))D^iT_1(u_n-u_m)\,dx\,dt\to 0 \text{ as } n,m\to\infty, \tag{4.56}$$

and thanks to (4.34), we have

$$\begin{aligned}
&\Big|\int_0^s\int_\Omega(g_n(x,t,u_n)-g_m(x,t,u_m))T_1(u_n-u_m)\,dx\,dt\Big|\\
&\qquad\leqslant\int_{Q_T}\big|g_n(x,t,u_n)-g_m(x,t,u_m)\big|\,dx\,dt\longrightarrow 0 \qquad \text{as} \quad n,m\to\infty,\\
&\Big|\int_0^s\int_\Omega\Big(\frac{1}{n}|u_n|^{p_0-2}u_n-\frac{1}{m}|u_m|^{p_0-2}u_m\Big)T_1(u_n-u_m)\,dx\,dt\Big|\\
&\qquad\leqslant\int_{Q_T}\Big|\frac{1}{n}|u_n|^{p_0-2}u_n-\frac{1}{m}|u_m|^{p_0-2}u_m\Big|\,dx\,dt\longrightarrow 0 \qquad \text{as} \quad n,m\to\infty,
\end{aligned}$$

and

$$\Big|\int_0^s\int_\Omega(f_n-f_m)\cdot T_1(u_n-u_m)\,dx\,dt\Big|\leqslant\int_{Q_T}|f_n-f_m|\,dx\,dt\longrightarrow 0 \qquad \text{as} \quad n,m\to\infty.$$

Since $\varphi_1(u_{0,n}-u_{0,m}) \to 0$ in $L^1(\Omega)$, then we conclude that

$$\int_\Omega \varphi_1(u_n(s)-u_m(s))\,dx \longrightarrow 0 \quad \text{as} \quad n,m\to\infty. \tag{4.57}$$

On the other hand, we have

$$\begin{aligned}\int_{\{|u_n-u_m|\leqslant 1\}} &|u_n(s)-u_m(s)|^2\,dx + \int_{\{|u_n-u_m|>1\}} |u_n(s)-u_m(s)|\,dx \\ &\leqslant 2\int_\Omega \varphi_1(u_n(s)-u_m(s))\,dx,\end{aligned} \tag{4.58}$$

and

$$\begin{aligned}\int_\Omega |u_n(s)-u_m(s)|\,dx &= \int_{\{|u_n-u_m|\leqslant 1\}} |u_n(s)-u_m(s)|\,dx + \int_{\{|u_n-u_m|>1\}} |u_n(s)-u_m(s)|\,dx \\ &\leqslant \Big(\int_{\{|u_n-u_m|\leqslant 1\}} |u_n(s)-u_m(s)|^2\,dx\Big)^{\frac12}\cdot(\operatorname{meas}(\Omega))^{\frac12} \\ &\quad + \int_{\{|u_n-u_m|>1\}} |u_n(s)-u_m(s)|\,dx.\end{aligned} \tag{4.59}$$

In view of (4.57)–(4.59), we deduce that

$$\int_\Omega |u_n(s)-u_m(s)|\,dx \longrightarrow 0 \quad \text{as} \quad m,n\to\infty. \tag{4.60}$$

Hence, u_n is a Cauchy sequence in $C([0,T];L^1(\Omega))$, thus $u\in C([0,T];L^1(\Omega))$ and for any $0\leqslant s\leqslant T$ we have $u_n(s)\to u(s)$ in $L^1(\Omega)$.

Step 7: Passage to the limit. Let $\psi\in L^{\vec p}(0,T;W_0^{1,\vec p}(\Omega))\cap L^\infty(Q_T)$, By taking $T_k(u_n-\psi)\in L^{\vec p}(0,T;W_0^{1,\vec p}(\Omega))$ as a test function in (4.2), we obtain

$$\begin{aligned}\int_0^T \Big\langle \frac{\partial u_n}{\partial t}, T_k(u_n-\psi)\Big\rangle\,dt &+ \sum_{i=1}^N \int_{Q_T} a_i(x,t,T_n(u_n),\nabla u_n)D^iT_k(u_n-\psi)\,dx\,dt \\ &+ \int_{Q_T} g_n(x,t,u_n)T_k(u_n-\psi)\,dx\,dt + \frac1n\int_{Q_T}|u_n|^{p_0-2}u_nT_k(u_n-\psi)\,dx\,dt \\ =& \int_{Q_T} f_nT_k(u_n-\psi)\,dx\,dt.\end{aligned} \tag{4.61}$$

For the first term on the left-hand side of (4.61), we have $\dfrac{\partial u_n}{\partial t}=\dfrac{\partial(u_n-\psi)}{\partial t}+\dfrac{\partial\psi}{\partial t}$, then

$$\begin{aligned}\int_0^T \Big\langle \frac{\partial u_n}{\partial t}, T_k(u_n-\psi)\Big\rangle\,dt &= \int_0^T \Big\langle \frac{\partial (u_n-\psi)}{\partial t}, T_k(u_n-\psi)\Big\rangle\,dt \\ &\quad + \int_0^T \Big\langle \frac{\partial \psi}{\partial t}, T_k(u_n-\psi)\Big\rangle\,dt \\ &= \int_\Omega \varphi_k(u_n-\psi)(T)\,dx - \int_\Omega \varphi_k(u_n-\psi)(0)\,dx \\ &\quad + \int_{Q_T}\frac{\partial\psi}{\partial t}T_k(u_n-\psi)\,dx\,dt.\end{aligned}$$

Since $u_n \to u$ in $C([0,T];L^1(\Omega))$, then $u_n(T) \to u(T)$ in $L^1(\Omega)$. It follows that

$$\int_\Omega \varphi_k(u_n-\psi)(0)\,dx \longrightarrow \int_\Omega \varphi_k(u_0-\psi(0))\,dx \quad \text{and}$$
$$\int_\Omega \varphi_k(u_n-\psi)(T)\,dx \longrightarrow \int_\Omega \varphi_k(u-\psi)(T)\,dx. \tag{4.62}$$

Now, we have $\frac{\partial\psi}{\partial t} \in L^{\vec{p}'}(0,T;W^{1,\vec{p}'}(\Omega))+L^1(Q_T)$, and since $T_k(u_n-\psi) \rightharpoonup T_k(u-\psi)$ in $L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$ and weak$-\star$ in $L^\infty(Q_T)$, then

$$\int_{Q_T} \frac{\partial\psi}{\partial t} T_k(u_n-\psi)\,dx\,dt \longrightarrow \int_{Q_T} \frac{\partial\psi}{\partial t} T_k(u-\psi)\,dx\,dt. \tag{4.63}$$

Concerning the second term on the left-hand side of (4.61), it's clear that $\{|u_n-\psi| \leqslant k\} \subseteq \{|u_n| \leqslant M = k+\|\psi\|_{L^\infty}\}$, and since $D^iT_M(u_n) \to D^iT_M(u)$ strongly in $L^{p_i}(Q_T)$, then in view of Fatou's Lemma, we obtain

$$\begin{aligned}
&\liminf_{n\to+\infty}\int_{Q_T} a_i(x,t,T_n(u_n),\nabla u_n)D^iT_k(u_n-\psi)\,dx\,dt\\
&=\liminf_{n\to+\infty}\int_0^T\int_{\{|u_n-\psi|\leqslant k\}} a_i(x,t,T_M(u_n),\nabla T_M(u_n))(D^iT_M(u_n)-D^i\psi)\,dx\,dt\\
&\geqslant \int_0^T\int_{\{|u-\psi|\leqslant k\}}(a_i(x,t,T_M(u),\nabla T_M(u))-a_i(x,t,T_M(u),\nabla\psi))(D^iT_M(u)-D^i\psi)\,dx\,dt\\
&\quad+\int_0^T\int_{\{|u-\psi|\leqslant k\}} a_i(x,t,T_M(u),\nabla\psi)(D^iT_M(u)-D^i\psi)\,dx\,dt\\
&=\int_{Q_T} a_i(x,t,u,\nabla u)D^iT_k(u-\psi)\,dx\,dt.
\end{aligned} \tag{4.64}$$

Also, since $T_k(u_n-\psi) \rightharpoonup T_k(u-\psi)$ weak$-\star$ in $L^\infty(Q_T)$, and thanks to (4.34), we have

$$\begin{aligned}
\int_{Q_T} g_n(x,t,u_n)T_k(u_n-\psi)\,dx\,dt+\frac{1}{n}\int_{Q_T}|u_n|^{p_0-2}u_nT_k(u_n-\psi)\,dx\,dt&\\
\to \int_{Q_T} g(x,t,u)T_k(u-\psi)\,dx\,dt,&
\end{aligned} \tag{4.65}$$

and

$$\int_{Q_T} f_nT_k(u_n-\psi)\,dx\,dt \longrightarrow \int_{Q_T} fT_k(u-\psi)\,dx\,dt. \tag{4.66}$$

By combining (4.61)–(4.66), we deduce that

$$\begin{aligned}
&\int_\Omega \varphi_k(u-\psi)(T)\,dx-\int_\Omega \varphi_k(u-\psi)(0)\,dx+\int_{Q_T}\frac{\partial\psi}{\partial t}T_k(u-\psi)\,dx\,dt\\
&\quad+\sum_{i=1}^N\int_{Q_T} a_i(x,t,u,\nabla u)D^iT_k(u-\psi)\,dx\,dt+\int_{Q_T} g(x,t,u)T_k(u-\psi)\,dx\,dt\\
&\leqslant \int_{Q_T} fT_k(u-\psi)\,dx\,dt,
\end{aligned}$$

which concludes our proof. □

5. Renormalized solutions

DEFINITION 5.1. Let $f \in L^1(Q_T)$ and $u_0 \in L^1(\Omega)$. A measurable function u is a renormalized solution of the anisotropic parabolic problem (3.7) if $T_k(u) \in L^{\vec{p}}(0,T; W_0^{1,\vec{p}}(\Omega))$ for all $k > 0$,

$$u \in C([0,T];L^1(\Omega)), \qquad \lim_{h\to\infty}\sum_{i=1}^{N}\int_{\{h\leqslant|u|\leqslant h+1\}} a_i(x,t,u,\nabla u)D^i u\,dx\,dt = 0,$$

and

$$\begin{cases} \displaystyle\int_{Q_T}\frac{dS(u)}{dt}\varphi\,dx\,dt + \sum_{i=1}^{N}\int_{Q_T} a_i(x,t,u,\nabla u)\cdot(S''(u)\varphi D^i u + S'(u)D^i\varphi)\,dx\,dt \\ \quad + \displaystyle\int_{Q_T} g(x,t,u)\,S'(u)\varphi\,dx\,dt = \int_{Q_T} fS'(u)\varphi\,dx\,dt, \end{cases} \tag{5.1}$$

for every function $\varphi \in L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega)) \cap L^\infty(Q_T)$ and any renormalization $S(\cdot) \in C^\infty(\mathbb{R})$ such that $\operatorname{supp} S'(\cdot) \subseteq [-M,M]$ for some constant $M > 0$.

THEOREM 5.1. *Let $f \in L^1(Q_T)$ and $u_0 \in L^1(\Omega)$. Under the assumptions (3.1)–(3.6), the entropy solution of the quasilinear anisotropic parabolic problem (3.7) is also a renormalized solution.*

Proof. We shall prove that every entropy solution u satisfies all the properties of renormalized solutions.

Indeed, in view of Theorem 4.1, there exists a subsequence $(u_n)_n$ of solutions for the approximate problems (4.2) such that $T_k(u_n)$ strongly converges to $T_k(u)$ in $L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$ for any $k > 0$, and satisfies

$$g_n(x,t,u_n) \longrightarrow g(x,t,u) \quad \text{and} \quad \frac{1}{n}|u_n|^{p_0-1} \longrightarrow 0 \quad \text{in} \quad L^1(Q_T).$$

Also, in view of Fatou's Lemma, we deduce that

$$\sum_{i=1}^{N}\int_{\{h\leqslant|u|<h+1\}} a_i(x,t,u,\nabla u)D^i u\,dx\,dt$$

$$\leqslant \liminf_{n\to\infty}\sum_{i=1}^{N}\int_{\{h\leqslant|u_n|<h+1\}} a_i(x,t,u_n,\nabla u_n)D^i u_n\,dx\,dt \to 0 \quad \text{as} \quad h\to\infty.$$

Now, we will show the equality (5.1).

Let $\varphi \in L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))\cap L^\infty(Q_T)$ and $S(\cdot)\in C^\infty(\mathbb{R})$, with $\operatorname{supp} S'(\cdot)\subset[-M,M]$ for some $M>0$. By taking $S'(u_n)\varphi$ a test function in (4.2), we obtain

$$\begin{aligned} &\int_0^T \Big\langle \frac{\partial u_n}{\partial t}, S'(u_n)\varphi \Big\rangle dt + \sum_{i=1}^{N}\int_{Q_T} a_i(x,t,T_n(u_n),\nabla u_n)D^i(S'(u_n)\varphi)\,dx\,dt \\ &\quad + \int_{Q_T} g_n(x,t,u_n)S'(u_n)\varphi\,dx\,dt + \frac{1}{n}\int_{Q_T}|u_n|^{p_0-2}u_nS'(u_n)\varphi\,dx\,dt \\ &= \int_{Q_T} f_nS'(u_n)\varphi\,dx\,dt. \end{aligned} \tag{5.2}$$

First, in view of (4.39), we have $\dfrac{\partial S(u_n)}{\partial t} \rightharpoonup \dfrac{\partial S(u)}{\partial t}$ weakly in $L^{\vec{p}'}(0,T;W^{-1,\vec{p}'}(\Omega))+L^1(Q_T)$, and then

$$\lim_{n\to\infty}\int_0^T \Big\langle \frac{\partial u_n}{\partial t}, S'(u_n)\varphi\Big\rangle\,dt = \lim_{n\to\infty}\int_{Q_T}\frac{\partial S(u_n)}{\partial t}\varphi\,dx\,dt = \int_{Q_T}\frac{\partial S(u)}{\partial t}\varphi\,dx\,dt. \tag{5.3}$$

Concerning the second term on the left-hand side of (5.2), we have

$$\begin{aligned}&\int_{Q_T} a_i(x,t,T_n(u_n),\nabla u_n)\cdot D^i(S'(u_n)\varphi)\,dx\,dt\\ &= \int_{Q_T} a_i(x,t,T_M(u_n),\nabla T_M(u_n))\cdot(S''(u_n)\varphi D^i T_M(u_n)+S'(u_n)D^i\varphi)\,dx\,dt.\end{aligned}$$

By (3.2) we have $a_i(x,t,T_M(u_n),\nabla T_M(u_n))$ is bounded in $(L^{p_i'}(Q_T))^N$, and $a_i(x,t,T_M(u_n),\nabla T_M(u_n)) \to a_i(x,t,T_M(u),\nabla T_M(u))$ a.e. in Q_T, it follows that

$$a_i(x,t,T_M(u_n),\nabla T_M(u_n)) \rightharpoonup a_i(x,t,T_M(u),\nabla T_M(u)) \quad \text{in } L^{p_i'}(Q_T),$$

and since

$$S''(u_n)\varphi D^i T_M(u_n)+S'(u_n)D^i\varphi \longrightarrow S''(u)\varphi D^i T_M(u)+S'(u)D^i\varphi \quad \text{in } L^{p_i}(Q_T),$$

we conclude that

$$\begin{aligned}&\lim_{n\to\infty}\int_{Q_T} a_i(x,t,T_M(u_n),\nabla T_M(u_n))(S''(u_n)\varphi D^i T_M(u_n)+S'(u_n)D^i\varphi)\,dx\,dt\\ &= \int_{Q_T} a_i(x,t,T_M(u),\nabla T_M(u))(S''(u)\varphi D^i T_M(u)+S'(u)D^i\varphi)\,dx\,dt\\ &= \int_{Q_T} a_i(x,t,u,\nabla u)(S''(u)\varphi D^i u+S'(u)D^i\varphi)\,dx\,dt.\end{aligned} \tag{5.4}$$

Moreover, since $S(u_n)\,\varphi \rightharpoonup S(u)\,\varphi$ weak $-\star$ in $L^\infty(\Omega)$, then

$$\int_{Q_T} g_n(x,t,u_n)S'(u_n)\,\varphi\,dx + \frac{1}{n}\int_{Q_T}|u_n|^{p_0-2}u_nS'(u_n)\,\varphi\,dx \longrightarrow \int_{Q_T} g(x,t,u)S'(u)\varphi\,dx, \tag{5.5}$$

and

$$\int_\Omega f_nS'(u_n)\,\varphi\,dx \longrightarrow \int_\Omega fS'(u)\,\varphi\,dx. \tag{5.6}$$

By combining (5.2)–(5.6), we deduce that

$$\begin{aligned}&\int_0^T\Big\langle\frac{\partial S(u)}{\partial t},\varphi\Big\rangle\,dt + \sum_{i=1}^N\int_{Q_T} a_i(x,t,u,\nabla u)\cdot(S''(u)\varphi D^i u+S'(u)D^i\varphi)\,dx\,dt\\ &\quad+\int_{Q_T} g(x,t,u)S'(u)\varphi\,dx\,dt = \int_{Q_T} fS'(u)\varphi\,dx\,dt.\end{aligned} \tag{5.7}$$

Therefore, u is a renormalized solution to problem (3.7). □

EXEMPLE 5.1. Let us consider the following nonlinear parabolic problem

$$\begin{cases} u_t + \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2} \frac{\partial u}{\partial x_i}\right) + \gamma|u|^{p_0-2}u = f & \text{in} \quad Q_T = \Omega \times (0,T), \\ u(x,t) = 0 & \text{on} \quad \Sigma_T = \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in} \quad \Omega, \end{cases} \tag{5.8}$$

where $u_0 \in L^1(\Omega)$, $\gamma > 0$ and $f \in L^1(Q_T)$. Note that the assumptions (3.1)–(3.6) hold true, then there exists at least one entropy solution. Moreover, this entropy solution is a renormalized solution with $|u|^{p_0-1} \in L^1(\Omega)$.

6. Appendix

Let $h > 0$, we set

$$E_{n,m} = \{|u_n - u_m| \leqslant 1\}, \quad H^-_{m,n} = \{|u_n| \leqslant h\} \cap \{|u_m| \leqslant h\}$$

and

$$H^+_{m,n} = \{|u_n| > h\} \cap \{|u_m| > h\}.$$

In view of Young's inequality, we have

$$\begin{aligned} &\Big|\int_0^s \int_\Omega \big(a_i(x,t,T_n(u_n),\nabla u_n) - a_i(x,t,T_m(u_m),\nabla u_m)\big) D^i T_1(u_n - u_m)\, dx\, dt\Big| \\ &= \Big|\int_0^s \int_\Omega \big(a_i(x,t,T_n(u_n),\nabla u_n) - a_i(x,t,T_m(u_m),\nabla u_m)\big)(D^i u_n - D^i u_m) \cdot \chi_{E_{n,m}}\, dx\, dt\Big| \\ &\leqslant \int_{H^-_{m,n}} \big|a_i(x,t,T_h(u_n),\nabla T_h(u_n)) - a_i(x,t,T_h(u_m),\nabla T_h(u_m))\big| \\ &\quad \times |D^i T_h(u_n) - D^i T_h(u_m)| \cdot \chi_{E_{n,m}}\, dx\, dt \\ &\quad + \int_{H^+_{m,n}} \big|a_i(x,t,T_n(u_n),\nabla u_n) - a_i(x,t,T_h(u_m),\nabla u_m)\big|^{p_i'} \cdot \chi_{E_{n,m}}\, dx\, dt \\ &\quad + \int_{H^+_{m,n}} |D^i u_n - D^i u_m|^{p_i} \cdot \chi_{E_{n,m}}\, dx\, dt. \end{aligned} \tag{6.1}$$

For the first term on the right-hand side of (6.1), since $D^i T_h(u_n)$ and $D^i T_h(u_m)$ converge strongly to $D^i T_h(u)$ in $L^{p_i}(Q_T)$, and

$$\big|a_i(x,t,T_h(u_n),\nabla T_h(u_n)) - a_i(x,t,T_h(u_m),\nabla T_h(u_m))\big| \quad \text{is bounded in} \quad L^{p_i'}(Q_T),$$

then

$$\begin{aligned} &\int_{H^+_{m,n}} \big|a_i(x,t,T_h(u_n),\nabla T_h(u_n)) - a_i(x,t,T_h(u_m),\nabla T_h(u_m))\big| \\ &\qquad \times |D^i T_h(u_n) - D^i T_h(u_m)| \cdot \chi_{E_{n,m}}\, dx\, dt \longrightarrow 0, \end{aligned} \tag{6.2}$$

as m and n tend to infinity.

Concerning the two last terms on the right-hand side of (6.1), we have

$$H^+_{m,n} \cap E_{n,m} \subseteq \{|u_n| > h\} \cap \{|u_n| - 1 \leqslant |u_m| \leqslant |u_n| + 1\},$$

and

$$H^+_{m,n} \cap E_{n,m} \subseteq \{|u_m| > h\} \cap \{|u_m| - 1 \leqslant |u_n| \leqslant |u_m| + 1\}.$$

Thanks to (3.2) and (4.27), we obtain

$$\begin{aligned}
&\int_{H^+_{m,n}} \left|a_i(x,t,T_n(u_n),\nabla u_n) - a_i(x,t,T_m(u_m),\nabla u_m)\right|^{p_i'} \cdot \chi_{E_{n,m}}\,dx\,dt \\
&\leqslant c_1 \int_{H^+_{m,n}} (2|K(x,t)|^{p_i'} + |T_n(u_n)|^{p_i} + |T_m(u_m)|^{p_i} + |D^i u_n|^{p_i} + |D^i u_m|^{p_i}) \cdot \chi_{E_{n,m}}\,dx\,dt \\
&\leqslant c_1 \int_{H^+_{m,n}} (2|K(x,t)|^{p_i'} + |T_n(u_n)|^{p_i} + |T_m(u_m)|^{p_i}) \cdot \chi_{E_{n,m}}\,dx\,dt \\
&\quad + c_1 \int_{\{|u_m|>h\}\cap\{|u_m|-1\leqslant|u_n|\leqslant|u_m|+1\}} |D^i u_n|^{p_i}\,dx\,dt \\
&\quad + c_1 \int_{\{|u_n|>h\}\cap\{|u_n|-1\leqslant|u_m|\leqslant|u_n|+1\}} |D^i u_m|^{p_i}\,dx\,dt \longrightarrow 0 \qquad \text{as} \quad h \to \infty.
\end{aligned} \tag{6.3}$$

Similarly, we prove that

$$\int_{H^+_{m,n}} |D^i u_n - D^i u_m|^{p_i} \cdot \chi_{\{|u_n - u_m| \leqslant 1\}}\,dx\,dt \longrightarrow 0 \quad \text{as} \quad h \to \infty. \tag{6.4}$$

By combining (6.2)–(6.4), we deduce that

$$\int_0^s \int_\Omega \big(a_i(x,t,T_n(u_n),\nabla u_n) - a_i(x,t,T_m(u_m),\nabla u_m)\big) D^i T_1(u_n - u_m)\,dx\,dt \longrightarrow 0 \tag{6.5}$$

as n, $m \to \infty$, then the convergence (4.56) is proved.

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