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Elliptic equations in weighted Sobolev spaces of infinite order with L^1 data

Mostafa Bendahmane,[a] Moussa Chrif[b] and Said El Manouni[a]*†

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We deal with the existence and uniqueness of weak solutions for a class of strongly nonlinear boundary value problems of higher order with L^1 data in anisotropic-weighted Sobolev spaces of infinite order.

Keywords: anisotropic-weighted Sobolev spaces; higher order; L^1 data; sign condition; existence and uniqueness results

1. Introduction

Let Ω be an open-bounded subset of $\mathbb{R}^N, N\geqslant 2$. For a positive integer m, let $W^{m,\vec{p}}(\Omega,\vec{\varpi})$ be the weighted anisotropic Sobolev space associated with the vector $\vec{p}$, with $\vec{p}$ denoting a vector of real numbers, i.e. $\vec{p}=\{p_\alpha,|\alpha|\leqslant m\}$, where p_α are real numbers and $\vec{\varpi}$ denote a vector of measurable positive functions, i.e. $\vec{\varpi}=\{\omega_\alpha(x),|\alpha|\leqslant m\}$, where $\omega_\alpha(x)$ are measurable positive functions for all multi-indices α.

Consider the following strongly nonlinear Dirichlet problem:

$$Au+g(x,u)=f \quad \text{in } \Omega \tag{1}$$

where A is a nonlinear elliptic operator of infinite order satisfying certain growth and coerciveness conditions. g is a nonlinear lower-order term having to fulfil only a sign condition $g(x,s)s\geqslant 0$, without assuming any growth conditions with respect to $|u|$ and f is given data.

Let us mention that, when f belongs to the dual space and A is a Leray–Lions operator, extensive attention has been paid to the investigation of boundary value problems of higher order associated with the equation of the type (1) in the setting of Sobolev and Orlicz–Sobolev spaces; we can cite for example [1] and [2] and the references therein. Some existence results are given under some conditions on the regularity of the domain (at least the segment property), since the techniques used are based on an approximation of the Hedberg type (cf. [3]).

In this context of Leray–Lions operators, in the L^1 case, recall that several papers have appeared in this direction in the setting of classical Sobolev spaces and a lot of articles have appeared (cf. [4–7]). Let us also mention that in the recent work [8], the authors proved the existence of solutions in the setting of anisotropic spaces of finite order. It is the purpose of this work to get the existence results in the setting of anisotropic-weighted Sobolev spaces for a class of nonlinear elliptic equations of infinite order of type (1) with L^1 data, which include as a special case problems involving Leray–Lions operators in the usual sense. More precisely, we will consider a nonlinear operator of infinite order of type

$$Au=\sum_{|\alpha|=0}^{\infty}(-1)^{|\alpha|}D^\alpha(A_\alpha(x,D^\gamma u)), \quad |\gamma|\leqslant|\alpha|$$

and prove the existence of weak solutions for the problem (1) in the case where the datum f is an element of $L^1(\Omega)$ in the setting of anisotropic-weighted Sobolev space of infinite order $W^\infty(a_\alpha,p_\alpha,\omega_\alpha)(\Omega)$ without assuming any monotonicity condition.

Finally, it would be interesting at this stage to refer the reader to the work of Dubinskiĭ [9] where the author proved, under hypothesis $(A_1)-(A_4)$ (see below), the existence of solution for the Dirichlet problem associated with the equation $Au=f$ and to the work [10] dealing with problems of finite and infinite order of type (1) in the setting of weighted anisotropic Sobolev spaces with data in dual.

[a] *Faculty of Sciences, Department of Mathematics, Al-Imam University, P. O. Box 90950, Riyadh 11623, Saudi Arabia*
[b] *Faculty of Sciences DM, University of Fès, B.P. 1796 Atlas-Fès, 30000, Morocco*
* *Correspondence to: Said El Manouni, Faculty of Sciences, Department of Mathematics, Al-Imam University, P. O. Box 90950, Riyadh 11623, Saudi Arabia.*
† *E-mail: manouni@hotmail.com*

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2. Preliminaries

Let Ω be a smooth bounded-domain in $\mathbb{R}^N$. Almost everywhere positive and locally integrable function $\omega:\Omega \longrightarrow \mathbb{R}$ will be called a weight. We shall denote by $L^p_\omega(\Omega)$ the set of all measurable functions u on Ω such that the norm

$$\|u\|_{L^p_\omega} \equiv \|u\|_{p,\omega} = \left(\int_\Omega |u|^p \omega(x)\,dx\right)^{1/p}, \quad 1\leqslant p<\infty$$

is finite. $L^p_\omega(\Omega)$ is also called weighted Lebesgue space.

Further $p_\alpha \geqslant 1$ are real numbers and ω_α are weight measurable functions for all multi-indices α. For a positive integer m, we define the following vectors:

$$\overrightarrow{p} = \{p_\alpha, |\alpha|\leqslant m\} \quad \text{and} \quad \overrightarrow{\varpi} = \{\omega_\alpha = \omega_\alpha(x), |\alpha|\leqslant m\}$$

where $\alpha = (\alpha_1, \ldots, \alpha_N)$ is a multi-index for differentiation and $|\alpha| = \sum_{i=1}^N \alpha_i$. The weighted anisotropic Sobolev space $W^{m,\overrightarrow{p}}(\Omega, \overrightarrow{\varpi})$ is defined as the collection of all functions $u \in L^1_{\text{loc}}(\Omega)$, having the generalized derivatives $D^\alpha u, |\alpha|\leqslant m$ with the finite norm

$$\|u\| = \sum_{|\alpha|=0}^{m} \|D^\alpha u\|_{p_\alpha,\omega_\alpha} \tag{2}$$

Here $D^\alpha = \partial^{|\alpha|}/(\partial x_1)^{\alpha_1} ::: (\partial x_N)^{\alpha_N}$. We denote by $C_0^\infty(\Omega)$ the space of all functions with compact support in Ω with continuous derivatives of arbitrary order.

We define the functional space $W_0^{m,\overrightarrow{p}}(\Omega, \overrightarrow{\varpi})$ as the closure of $C_0^\infty(\Omega)$ in $W^{m,\overrightarrow{p}}(\Omega, \overrightarrow{\varpi})$ with respect to the norm (2). Note that $C_0^\infty(\Omega)$ is dense in $W_0^{m,\overrightarrow{p}}(\Omega, \overrightarrow{\varpi})$. By an adapted method as that of Adams [11] and by constructing an isometric isomorphism from $W^{m,\overrightarrow{p}}(\Omega, \overrightarrow{\varpi})$ into $\prod_{|\alpha|=0}^m L^{p_\alpha}_{\omega_\alpha}(\Omega)$, we can show that $W_0^{m,\overrightarrow{p}}(\Omega, \omega)$ is separable and reflexive if $1\leqslant p_\alpha<\infty$ and $1<p_\alpha<\infty$, respectively, for all $|\alpha|\leqslant m$. For $p_\alpha>1, |\alpha|\leqslant m, W^{-m,\overrightarrow{p}'}(\Omega, \overrightarrow{\varpi}')$ designs its dual where $\overrightarrow{p}'$ is the conjugate of $\overrightarrow{p}$, i.e. $p'_\alpha = p_\alpha/(p_\alpha - 1)$ and $\overrightarrow{\varpi}' = \{{\omega_\alpha}' = {\omega_\alpha}^{1-p'_\alpha}, |\alpha|\leqslant m\}$. By the same techniques used in [11] in the framework of classical Sobolev spaces, we can easily show that the dual $W^{-m,\overrightarrow{p}'}(\Omega, \overrightarrow{\varpi}')$ can be identified to the set

$$\mathcal{I}_{m,\overrightarrow{p}} = \left\{ T \in D'(\Omega) / \exists v \in L^{\overrightarrow{p}'}_{\overrightarrow{\varpi}'}(\Omega) : T = \sum_{0\leqslant|\alpha|\leqslant m} (-1)^{|\alpha|} D^\alpha v_\alpha \right\}$$

with

$$L^{\overrightarrow{p}'}_{\overrightarrow{\varpi}'}(\Omega) = \prod_{0\leqslant|\alpha|\leqslant m} L^{p'_\alpha}_{\omega'_\alpha}(\Omega)$$

Let $a_\alpha>0$ be a sequence of real numbers for all multi-indices α. The weighted Sobolev space of infinite order is the functional space defined by

$$W^\infty(a_\alpha, p_\alpha, \omega_\alpha)(\Omega) = \left\{ u \in C^\infty(\Omega) : \rho(u) = \sum_{|\alpha|=0}^{\infty} a_\alpha \|D^\alpha u\|^{p_\alpha}_{p_\alpha,\omega\alpha} < \infty \right\}$$

Since we shall deal with the Dirichlet problem, we shall use the functional space $W_0^\infty(a_\alpha, p_\alpha, \omega_\alpha)(\Omega)$ defined by

$$W_0^\infty(a_\alpha, p_\alpha, \omega_\alpha)(\Omega) = \left\{ u \in C_0^\infty(\Omega) : \rho(u) = \sum_{|\alpha|=0}^{\infty} a_\alpha \|D^\alpha u\|^{p_\alpha}_{p_\alpha,\omega_\alpha} < \infty \right\}$$

In contrast with the finite order Sobolev space, the very first question, which arises in the study of the space $W^\infty(a_\alpha, p_\alpha, \omega_\alpha)(\Omega)$, is the question of their nontriviality (or nonemptiness), i.e. the question of the existence of a function $u(x)$ such that $\rho(u)<\infty$. It turns out that the answer to this question depends not only on the given parameters a_α, p_α of the space $W^\infty(a_\alpha, p_\alpha, \omega_\alpha)(\Omega)$, but also on the domain Ω. We will use the following definition that, on the basis of [9], can be equivalently stated as follows:

Definition 2.1
The space $W_0^\infty(a_\alpha, p_\alpha, \omega_\alpha)(\Omega)$ is called nontrivial if it contains at least one function that is not identically equal to zero, i.e. there is a function $u(x) \in C_0^\infty(\Omega)$ such that $\rho(u)<\infty$.

The dual space of $W_0^\infty(a_\alpha, p_\alpha, \omega_\alpha)(\Omega)$ is defined as follows:

$$W^{-\infty}(a_\alpha, p'_\alpha, \omega'_\alpha)(\Omega) = \left\{ h \in D'(\Omega) : h = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha D^\alpha h_\alpha, \rho'(h) = \sum_{|\alpha|=0}^{\infty} a_\alpha \|h_\alpha\|^{p'_\alpha}_{p'_\alpha,\omega'_\alpha} < \infty \right\}$$

where $h_\alpha \in L^{p'_\alpha}_{\omega'_\alpha}(\Omega)$ and p'_α is the conjugate of p_α, i.e. $p'_\alpha = p_\alpha/(p_\alpha - 1)$.

By the definition, the duality pairing of the space $W^{-\infty}(a_\alpha, p'_\alpha, \omega'_\alpha)(\Omega)$ and $W_0^{\infty}(a_\alpha, p_\alpha, \omega_\alpha)(\Omega)$ is given by the relation

$$\langle h, v\rangle = \sum_{|\alpha|=0}^{\infty} a_\alpha \int_\Omega h_\alpha(x) D^\alpha v_\alpha(x)\,dx$$

which, as it is not difficult to verify, is correct (for more details about classical weighted Sobolev spaces, we refer the reader to [12] and for non-weighted anisotropic Sobolev spaces of finite and infinite order, see [9, 13]).

Let s be a real positive number. We denote by $E(s)$ the integer part of s and denote $\underline{p} = \min\{p_\alpha, |\alpha| \leqslant m\}$. We need the anisotropic Sobolev embedding result.

Lemma 2.1
Let Ω be a smooth bounded open subset of $\mathbb{R}^N$, and suppose that $\inf \omega_\alpha > 0$ a.e. in Ω for all $|\alpha| \leqslant m$.

If $m\underline{p} < N$, then $W_0^{m,\overrightarrow{p}}(\Omega, \overrightarrow{\omega}) \subset L^q(\Omega)$ for all $q \in [\underline{p}, p^*[$ with $\frac{1}{p^*} = \frac{1}{\underline{p}} - \frac{m}{N}$.

If $m\underline{p} = N$, then $W_0^{m,\overrightarrow{p}}(\Omega, \overrightarrow{\omega}) \subset L^q(\Omega)$ for all $q \in [\underline{p}, +\infty[$.

If $m\underline{p} > N$, then $W_0^{m,\overrightarrow{p}}(\Omega, \overrightarrow{\omega}) \subset L^\infty(\Omega) \cap C^k(\overline{\Omega})$ where $k = E(m - \frac{N}{\underline{p}})$.

Moreover, the embeddings are compacts.

The proof follows immediately from the corresponding embedding theorems in the isotropic case (see Adams [11]) and by using the fact that

$$W_0^{m,\overrightarrow{p}}(\Omega, \overrightarrow{\omega}) \subset W_0^{m,\overrightarrow{p}}(\Omega) \subset W_0^{m,\underline{p}}(\Omega)$$

3. Definitions and main results

In this section we formulate and prove the main result of the paper.

Definition 3.1 (Brézis [14])
Let Y be a reflexive Banach space. A bounded mapping B from Y to Y^* is called *pseudo-monotone* if for any sequence $u_n \in Y$ with $u_n * u$ weakly in Y and $\limsup_{n\longrightarrow\infty} \langle Bu_n, u_n - u\rangle \leqslant 0$, one has

$$\lim_{n\longrightarrow\infty}\inf \langle Bu_n, u_n - v\rangle \geqslant \langle Bu, u - v\rangle \quad \text{for all } v \in Y$$

We denote by λ_α the number of multi-indices γ such that $|\gamma| \leqslant |\alpha|$. Let A be an operator of infinite order defined by

$$A(u) = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} D^\alpha A_\alpha(x, D^\gamma u), \quad |\gamma| \leqslant |\alpha|$$

with $A_\alpha : \Omega \times \mathbb{R}^{\lambda_\alpha} \mapsto \mathbb{R}$ is a real function. Let us now formulate the assumptions

(A_1) $A_\alpha(x, \xi_\gamma)$ is a Carathéodory function for all $\alpha, |\gamma| \leqslant |\alpha|$.

(A_2) For a.e. $x \in \Omega$, all $m \in \mathbb{N}^*$, all $\xi_\gamma, \eta_\alpha, |\gamma| \leqslant |\alpha|$ and some constant $c_0 > 0$, we assume that

$$\left| \sum_{|\alpha|=0}^{m} A_\alpha(x, \xi_\gamma)\eta_\alpha \right| \leqslant c_0 \sum_{|\alpha|=0}^{m} a_\alpha \omega_\alpha |\xi_\alpha|^{p_\alpha - 1} |\eta_\alpha|$$

where $a_\alpha \geqslant 0, p_\alpha > 1$ are real numbers for all multi-indices α and for all bounded sequence $(p_\alpha)_\alpha$. Here, ω_α are weight functions satisfying $\inf w_\alpha > 0$ a.e. in Ω for all $|\alpha| \leqslant m$.

(A_3) There exist constants $c_1 > 0, c_2 \geqslant 0$ such that for all $m \in \mathbb{N}^*$, for all $\xi_\gamma, \xi_\alpha; |\gamma| \leqslant |\alpha|$, we have

$$\sum_{|\alpha|=0}^{m} A_\alpha(x, \xi_\gamma)\xi_\alpha \geqslant c_1 \sum_{|\alpha|=0}^{m} a_\alpha \omega_\alpha |\xi_\alpha|^{p_\alpha} - c_2$$

(A_4) The space $W_0^{\infty}(a_\alpha, p_\alpha, \omega_\alpha)(\Omega)$ is nontrivial.

(G_1) The function $g : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ is of Carathéodory type such that, for all $\delta > 0$,

$$\sup_{|u| \leqslant \delta} |g(x, u)| \leqslant h_\delta(x) \in L^1(\Omega)$$

(G_2) We assume the 'sign condition' $g(x, u)u \geqslant 0$, for a.e. $x \in \Omega$ and all $u \in \mathbb{R}$.

As regard to the second member, we assume that

$$f \in L^1(\Omega)$$

and we shall prove the following existence theorem.

Theorem 3.1
Assume that the assumptions $(A_1)-(A_4)$, (G_1) and (G_2) hold, then for all $f\in L^1(\Omega)$, there exists at least one solution $u\in W_0^{\infty}(a_\alpha,p_\alpha,\omega_\alpha)(\Omega)$ of (1) in the following sense:

$$\begin{cases} g(x,u)\in L^1(\Omega), \quad g(x,u)u\in L^1(\Omega) \\ \langle Au,v\rangle+\int_\Omega g(x,u)v\,dx=\langle f,v\rangle \quad \text{for all } v\in W_0^{\infty}(a_\alpha,p_\alpha,\omega_\alpha)(\Omega) \end{cases}$$

Proof
We proceed by steps in order to prove our result.

`Step (1)`: The approximate problem.

Let $\varphi\in C_0^\infty(\mathbb{R}^N)$ such that $0<\varphi(x)<1$ and $\varphi(x)=1$ for x close to 0. Set

$$f_n(x)=\varphi\left(\frac{x}{n}\right)T_n f(x)$$

where T_n is the usual truncation given by

$$T_n\xi=\begin{cases}\xi & \text{if } |\xi|<n\\ \dfrac{n\xi}{|\xi|} & \text{if } |\xi|\geqslant n\end{cases}$$

It is clear that $|f_n|\leqslant n$ for a.e. $x\in\Omega$. Thus, it follows that $f_n\in L^\infty(\Omega)$. Using Lebesgue's dominated convergence theorem, since $f_n\longrightarrow f$ a.e. $x\in\Omega$ and $|f_n|\leqslant|f|\in L^1(\Omega)$, we conclude that f_n strongly converges to f in $L^1(\Omega)$.

Let $n\in\mathbb{N}^*$ is sufficiently large. Define the operator A_{2n+2} of order $2n+2$ by

$$A_{2n+2}(u)=\sum_{|\alpha|=n+1}(-1)^{n+1}c_\alpha D^{2\alpha}u+\sum_{|\alpha|=0}^{n}(-1)^{|\alpha|}D^\alpha A_\alpha(x,D^\gamma u),\quad |\gamma|\leqslant|\alpha|$$

where c_α are constants. Note that in the case of such an operator A_{2n+2} without weight, it was shown by Dubinskiĭ [9] that c_α are constants small enough such that they fulfil the conditions of the following lemma introduced in [9]. □

Lemma 3.1 (*Dubinskiĭ* [9])
For all nontrivial space $W_0^\infty(a_\alpha,p_\alpha)(\Omega)$, there exists a nontrivial space $W_0^\infty(c_\alpha,2)(\Omega)$ such that $W_0^\infty(a_\alpha,p_\alpha)(\Omega)\subset W_0^\infty(c_\alpha,2)(\Omega)$.

In our case, since $\inf\omega_\alpha>0$ a.e. in Ω for all $|\alpha|\leqslant m$, then $W_0^\infty(a_\alpha,p_\alpha,\omega_\alpha)(\Omega)$ is included in a space of infinite order W introduced by Dubinskiĭ [9] and which guarantees the existence of a nontrivial space $W_0^\infty(c_\alpha,2)(\Omega)$ such that $W_0^\infty(a_\alpha,p_\alpha,\omega_\alpha)(\Omega)\subset W\subset W_0^\infty(c_\alpha,2)(\Omega)$.

Now, the operator A_{2n+2} is clearly monotone since the term of higher order of derivation is linear and satisfies the monotonicity condition (and then pseudo-monotone); this follows from the result of [15]. Moreover from assumptions (A_1), (A_2) and (A_3), we deduce that A_{2n+2} satisfies the growth, the coerciveness and the monotonicity conditions. Thanks to ([10], Theorem 3.1), it follows from the theory of pseudo-monotone operators that there exists at least one solution $u_n\in W_0^{n+1,\vec{p}}(\Omega,\vec{\varpi})$ of the following approximate problem:

$$\langle A_{2n+2}(u_n),v\rangle+\int_\Omega g(x,u_n)v\,dx=\langle f_n,v\rangle,\quad v\in W_0^{n+1,\vec{p}}(\Omega,\vec{\varpi}) \tag{3}$$

Remark 3.1
Note that since n is taken sufficiently large, we can assume that $(n+1)\underline{p}>N$, then we have $W_0^{n+1,\vec{p}}(\Omega,\vec{\varpi})\subset L^\infty(\Omega)$. This implies that $g(x,u_n)\in L^1(\Omega)$ since

$$|g(x,u_n)|\leqslant h_{\|u_n\|_\infty}(x)\in L^1(\Omega)$$

and as a consequence, also $g(x,u_n)u_n$ belongs to $L^1(\Omega)$. On the other hand, using duality arguments, we deduce that $L^1(\Omega)\subset W^{-(n+1),\vec{p}\,'}(\Omega,\vec{\varpi}\,')$ and the case $f\in L^1(\Omega)$ may be considered as dual case (see [10]).

`Step (2)`: A priori estimates.

By choosing $v=u_n$ as test function in (P_n), we have

$$\sum_{|\alpha|=n+1}c_\alpha\int_\Omega|D^\alpha u_n|^2\,dx+\sum_{|\alpha|=0}^{n}\int_\Omega A_\alpha(x,D^\gamma u_n)D^\alpha u_n\,dx\leqslant\int_\Omega f_n u_n\,dx$$

Now, in view of (A_3), (G_2), the Hölder inequality and by using the fact that $|f_n|\leqslant|f|$, we get the estimates

$$\sum_{|\alpha|=n+1}c_\alpha\|D^\alpha u_n\|_2^2+\sum_{|\alpha|=0}^{n}a_\alpha\|D^\alpha u_n\|_{p_\alpha,\omega_\alpha}^{p_\alpha}\leqslant K \tag{4}$$

$$\int_\Omega g(x,u_n)u_n\,dx\leqslant K \tag{5}$$

for some constant $K=K(f)>0$. By considering $a_\alpha=c_\alpha, p_\alpha=2$ and $\omega_\alpha=1$ for $|\alpha|=n+1$, the estimate (4) is equivalent to

$$\sum_{|\alpha|=0}^{n+1} a_\alpha \|D^\alpha u_n\|_{p_\alpha,\omega_\alpha}^{p_\alpha} \leqslant K \tag{6}$$

Consequently, according to the definition of weighted anisotropic Sobolev spaces of finite order given above, we obtain

$$\|u_n\|_{W_0^{n+1,\overrightarrow{p}}(\Omega,\overrightarrow{\omega})} \leqslant K \tag{7}$$

Then, in a similar way as in [9, 16], via a diagonalization process, there exists a subsequence, still denoted by u_n, which converges uniformly to an element $u\in C_0^\infty(\Omega)$; also for all derivatives $D^\alpha u_n \to D^\alpha u$ holds.

`Step (3)`: Convergence of the approximate problem (P_n).

There exists a solution u_n of problem (P_n), $n=1,2,\ldots$. Then by passing to the limit, we have

$$\lim_{n\to+\infty}\langle A_{2n+2}(u_n), v\rangle + \lim_{n\to+\infty}\int_\Omega g(x,u_n)v\,dx = \lim_{n\to+\infty}\langle f_n, v\rangle$$

for $v\in W_0^\infty(a_\alpha,p_\alpha,\omega_\alpha)(\Omega)$. Since $f_n \longrightarrow f$ strongly in $L^1(\Omega)$, it is clear that

$$\lim_{n\to+\infty}\langle f_n, v\rangle = \langle f, v\rangle$$

for all $v\in W_0^\infty(a_\alpha,p_\alpha,\omega_\alpha)(\Omega)$.

Now, we shall prove that

$$\lim_{n\to+\infty}\langle A_{2n+2}(u_n), v\rangle = \langle Au, v\rangle \quad \text{for all } v\in W_0^\infty(a_\alpha,p_\alpha,\omega_\alpha)(\Omega)$$

In fact, let n_0 be a fix number sufficiently large ($n>n_0$) and let $v\in W_0^\infty(a_\alpha,p_\alpha,\omega_\alpha)(\Omega)$. Set $\langle A(u)-A_{2n+2}(u_n), v\rangle = I_1+I_2+I_3$, where

$$I_1 = \sum_{|\alpha|=0}^{n_0}\langle A_\alpha(x,D^\gamma u)-A_\alpha(x,D^\gamma u_n), D^\alpha v\rangle$$

$$I_2 = \sum_{|\alpha|=n_0+1}^{\infty}\langle A_\alpha(x,D^\gamma u), D^\alpha v\rangle$$

$$I_3 = -\sum_{|\alpha|=n_0+1}^{n+1}\langle A_\alpha(x,D^\gamma u_n), D^\alpha v\rangle$$

or in another form

$$I_3 = -\sum_{|\alpha|=n_0+1}^{n}\langle A_\alpha(x,D^\gamma u_n), D^\alpha v\rangle - \sum_{|\alpha|=n+1} c_\alpha\langle D^\alpha u_n, D^\alpha v\rangle$$

with $A_\alpha(x,\xi_\gamma)=c_\alpha\xi_\alpha, p_\alpha=2, w_\alpha=1$ and $c_\alpha\geqslant 0$ for $|\alpha|=n+1$.

We will pass to the limit as $n\longrightarrow+\infty$ to prove that I_1, I_2 and I_3 tend to 0. Starting with I_1, we have $I_1\to 0$ since $A_\alpha(x,\xi_\gamma)$ is of Carathéodory type. The term I_2 is the remainder of a convergent series, hence $I_2\to 0$. For what concerns I_3, in view of (A_2) and Hölder inequality we have

$$\begin{aligned}\left|\sum_{|\alpha|=n_0+1}^{n+1}\langle A_\alpha(x,D^\gamma u_n), D^\alpha v\rangle\right| &\leqslant \sum_{|\alpha|=n_0+1}^{n+1}|\langle A_\alpha(x,D^\gamma u_n), D^\alpha v\rangle| \\ &\leqslant c_0\sum_{|\alpha|=n_0+1}^{n+1} a_\alpha\int_\Omega \omega_\alpha |D^\alpha u_n|^{p_\alpha-1}|D^\alpha v|\,dx \\ &= c_0\sum_{|\alpha|=n_0+1}^{n+1} a_\alpha\int_\Omega \omega_\alpha^{1/p'_\alpha}|D^\alpha u_n|^{p_\alpha-1}\omega_\alpha^{1/p_\alpha}|D^\alpha v|\,dx \\ &\leqslant c_0\sum_{|\alpha|=n_0+1}^{n+1} a_\alpha\|D^\alpha u_n\|_{p_\alpha,\omega_\alpha}^{p_\alpha-1}\|D^\alpha v\|_{p_\alpha,\omega_\alpha}\end{aligned}$$

Therefore for all $\varepsilon>0$, there exists $k(\varepsilon)>0$ (see [17, p. 56]) such that

$$\begin{aligned}\left|\sum_{|\alpha|=n_0+1}^{n+1}\langle A_\alpha(x,D^\gamma u_n), D^\alpha v\rangle\right| &\leqslant \varepsilon c_0\sum_{|\alpha|=n_0+1}^{n+1} a_\alpha\|D^\alpha u_n\|_{p_\alpha,\omega_\alpha}^{p_\alpha} + c_0 k(\varepsilon)\sum_{|\alpha|=n_0+1}^{n+1} a_\alpha\|D^\alpha v\|_{p_\alpha,\omega_\alpha}^{p_\alpha} \\ &\leqslant \varepsilon c_0 K + c_0 k(\varepsilon)\sum_{|\alpha|=n_0+1}^{\infty} a_\alpha\|D^\alpha v\|_{p_\alpha,\omega_\alpha}^{p_\alpha}\end{aligned}$$

where K is the constant given in the estimate (4). Since the sequence (p_α) is bounded and $\sum_{|\alpha|=n_0+1}^{\infty} a_\alpha \|D^\alpha v\|_{p_\alpha,\omega_\alpha}^{p_\alpha}$ is the remainder of a convergent series, therefore $I_3 \to 0$ holds. Hence $\langle A_{2n+2}(u_n), v\rangle \to \langle A(u), v\rangle$ as $n\to+\infty$ for all $v\in W_0^\infty(a_\alpha, p_\alpha, \omega_\alpha)(\Omega)$.

It remains to show, for our purposes, that

$$\lim_{n\to+\infty}\int_\Omega g(x,u_n)v\,dx = \int_\Omega g(x,u)v\,dx$$

for $v\in W_0^\infty(a_\alpha,p_\alpha,\omega_\alpha)(\Omega)$. Indeed, we have $u_n\to u$ uniformly in Ω, hence $g(x,u_n)\to g(x,u)$ for a.e. $x\in\Omega$. In view of (5), we deduce by the Fatou's lemma that as $n\to\infty$

$$\int_\Omega g(x,u)u\,dx \leqslant \lim_{n\to+\infty}\int_\Omega g(x,u_n)u_n\,dx \leqslant K$$

this implies $g(x,u)u\in L^1(\Omega)$. On the other hand, let $\delta>0$, since $|g(x,t)|\delta\leqslant|g(x,t)t|$ and then $|g(x,t)|\leqslant\delta^{-1}|g(x,t)t|$ for $|t|\geqslant\delta$, we have

$$|g(x,u_n)| \leqslant \sup_{|t|\leqslant\delta}|g(x,t)| + \delta^{-1}|g(x,u_n)\cdot u_n|$$

$$\leqslant h_\delta(x) + \delta^{-1}|g(x,u_n)u_n|$$

On the other hand, we have

$$\int_E |g(x,u_n)|\,dx \leqslant \int_E h_\delta(x)\,dx + \delta^{-1}K$$

for some measurable subset E of Ω and for some $\varepsilon>0$. Here, K is the constant of (5) that is independent of n. For $|E|$ sufficiently small and $\delta=2K/\varepsilon$, we obtain $\int_E|g(x,u_n)|\,dx<\varepsilon$. Then, we get by using Vitali's theorem $g(x,u_n)\to g(x,u)$ in $L^1(\Omega)$. Hence it follows that $g(x,u)\in L^1(\Omega)$.

By passing to the limit, we obtain

$$\langle Au, v\rangle + \int_\Omega g(x,u)v\,dx = \langle f, v\rangle \quad \text{for all} \quad v\in W_0^\infty(a_\alpha,p_\alpha,\omega_\alpha)(\Omega)$$

Consequently,

$$\begin{cases} g(x,u)\in L^1(\Omega), \quad g(x,u)u\in L^1(\Omega) \\ \langle Au, v\rangle + \int_\Omega g(x,u)v\,dx = \langle f,v\rangle \quad \text{for all} \quad v\in W_0^\infty(a_\alpha,p_\alpha,\omega_\alpha)(\Omega) \end{cases}$$

This completes the proof. □

Remark 3.2
Note that the existence result is given without monotonicity condition on the operator.

4. Uniqueness

The purpose of this section is to prove uniqueness of weak solutions to our anisotropic elliptic equation with infinite order, thereby completing the well-posedness analysis. We make an additional assumption on A_α: There exists a constant $c_A>0$ such that for all $\xi_\gamma^1, \xi_\gamma^2, \xi_\alpha^1, \xi_\alpha^2, |\gamma|\leqslant|\alpha|$, we have

$$\sum_{|\alpha|=0}^{\infty}(A_\alpha(x,\xi_\gamma^1)-A_\alpha(x,\xi_\gamma^2))\cdot(\xi_\alpha^1-\xi_\alpha^2) > c_A\sum_{|\alpha|=0}^{\infty} a_\alpha\omega_\alpha|\xi_\alpha^1-\xi_\alpha^2|^{p_\alpha} \tag{8}$$

Theorem 4.1
Suppose the conditions $(A_1)-(A_4), (G_1), (G_2)$ and (8) are fulfilled. Furthermore, if g is nondecreasing in u, then the weak solution of problem (1) is unique.

Proof
Let u_1 and u_2 be two weak solutions to the problem (1). According to Theorem 3.1, the following equations hold for all test functions $\varphi\in W_0^\infty(a_\alpha,p_\alpha,\omega_\alpha)(\Omega)$:

$$\langle Au_1 - Au_2, \varphi\rangle + \int_\Omega (g(x,u_1)-g(x,u_2))\varphi\,dx = 0 \tag{9}$$

We utilize $\varphi = u_1-u_2$ in (9), (G_2) and (8) to obtain

$$\sum_{|\alpha|=0}^{+\infty} a_\alpha\int_\Omega |D^\alpha u_1 - D^\alpha u_2|^{p_\alpha}\omega_\alpha\,dx \leqslant 0$$

This implies

$$\int_\Omega |u_1-u_2|^{p_0}\,dx=0$$

with $p_\alpha=p_0$ for $|\alpha|=0$, ensuring the uniqueness result. □

Examples:

1. Let us consider the operator

$$Au=\sum_{|\alpha|=0}^{\infty}(-1)^{|\alpha|}D^\alpha(a_\alpha\omega_\alpha|D^\alpha u|^{p_\alpha-2}D^\alpha u)$$

   where $a_\alpha\geqslant 0, p_\alpha>1$ are real numbers and ω_α are weight functions such that the space $W_0^\infty(a_\alpha,p_\alpha,\omega_\alpha)(\Omega)$ is nontrivial. We can easily show that the conditions (A_1), (A_2) and (A_3) are satisfied.
2. A simple example of a function g satisfying the conditions (G_1) and (G_2) of Theorem 3.1 is given by $g(x,s)=\mathrm{sgn}(s)\exp(s)h(x)$, where $h\in L^1(\Omega), h(x)\geqslant 0$ a.e.

5. Application

Let ω be a weight, that is, almost everywhere positive and locally integrable function on Ω. Consider the following class of strongly nonlinear Dirichlet problem:

$$B(u)+u|u|^{r+1}h(x)=f \quad \text{in } \Omega \tag{10}$$

where $r>0$ is a real number, $h\in L^1(\Omega)$ with $h(x)\geqslant 0$ a.e $x\in\Omega$ and the operator B is defined as

$$B(u)=(-\sqrt{I+\omega\Delta})u$$

Our technique here consists to exploit certain result in the setting of functional spaces of infinite order. Thus as in [9] and [13], we can write the operator B as follows:

$$B(u)=(-\sqrt{I+\omega\Delta})u=\sum_{k=0}^{\infty}a_k\omega_k(-\Delta)^k u \tag{11}$$

where $a_k>0$ are real numbers and $\omega_k=\omega^k$ are weight functions for $k=0,1,\ldots$, which guarantee the nontriviality of the corresponding functional space (for more details see [9]). The characteristic function of our operator is defined by

$$\varphi(x)=\sum_k^{\infty}a_k\omega_k\xi^{2k} \quad \text{with } \xi^2=\xi_1^2+\cdots+\xi_N^2$$

and the corresponding space of the present strongly nonlinear Dirichlet problem is

$$W_0^\infty(a_k,2,\omega_k)(\Omega)=\left\{u\in C_0^\infty(\Omega):\rho(u)=\sum_{k=0}^{\infty}a_k\|\nabla^k u\|_{2,\omega_k}^2<\infty\right\}$$

Definition 5.1
A function $u\in W_0^\infty(a_k,2,\omega_k)(\Omega)$ is a solution of the Dirichlet problem (10), if for any $v\in W_0^\infty(a_k,2,\omega_k)(\Omega)$ the equality

$$\sum_{k=0}^{\infty}a_k\int_\Omega\omega_k\nabla^k u\nabla^k v\,dx+\int_\Omega u|u|^{r+1}h(x)v\,dx=\langle f,v\rangle$$

is valid.

The explicit form of the operator B given in (11) shows simply that it satisfies the assumptions (A_1), (A_2) and (A_3). Now, taking into account that B is monotone and the term $u|u|^{r+1}h(x)$ fulfils the sign condition, then by the same argument as in the proof of Theorem 3.1 we prove the following existence and uniqueness results.

Theorem 5.2
For all $f\in L^1(\Omega)$, there exists a unique solution $u\in W_0^\infty(a_k,2,\omega_k)(\Omega)$ such that

$$\langle Bu,v\rangle+\int_\Omega u|u|^{r+1}h(x)v\,dx=\langle f,v\rangle \quad \text{for all } v\in W_0^\infty(a_\alpha,2,\omega_k)(\Omega)$$

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