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Anisotropic equations in weighted Sobolev spaces of higher order

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Abstract We consider the strongly nonlinear boundary value problem,

$$Au + g(x, u) = f$$

where A is an elliptic operator of finite or infinite order. We introduce anisotropic weighted Sobolev spaces and we show under a certain sign condition of the Carathéodory function g without assuming any growth restrictions, the existence of the weak solutions.

Keywords Anisotropic weighted Sobolev spaces · Higher order · Finite and infinite orders · Monotonicity condition · Sign condition · Existence results

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1 Introduction

Let Ω be an open smooth bounded subset of $\mathbb{R}^N$, $N \geq 2$. Consider the following strongly nonlinear Dirichlet problem

$$Au + g(x, u) = f \quad \text{in } \Omega, \tag{1}$$

where A is a nonlinear elliptic operator satisfying certain growth and coerciveness conditions. g is a nonlinear lower order term having to fulfil only a sign condition $g(x, s)s \geq 0$, without assuming any growth conditions with respect to $|u|$ and f belongs to the dual space.

For a positive integer m, let $W^{m,\overrightarrow{p}}(\Omega, \overrightarrow{\omega})$ be the weighted anisotropic Sobolev space associated to the vector $\overrightarrow{p}$, with $\overrightarrow{p}$ denoting a vector of real numbers, i.e., $\overrightarrow{p} = \{p_\alpha, |\alpha| \leq m\}$, where p_α are real numbers and $\overrightarrow{\omega}$ denoting a vector of measurable positive functions, i.e., $\overrightarrow{\omega} = \{\omega_\alpha(x), |\alpha| \leq m\}$, where $\omega_\alpha(x)$ are measurable positive functions for all multi-indices α.

It is well know that if A is a Leray–Lions operator, several studies have been appeared (cf. [2,3,6,11]). In particular, Eq. 1 is the problem considered by Webb [11] in the classic Sobolev space $W_0^{m,p}(\Omega)(m \geq 1,\ 1 < p < \infty)$. Other existence results are proved, see, for example, Brézis and Browder [6] in the setting of Sobolev space and Benkirane and Gossez [3] in the setting of Sobolev–Orlicz space. In this context of Leray–Lions operators, let us recall that the results given have been proved under some conditions on the regularity of Ω (at least the segment property), since the techniques used are based on an approximation of the Hedberg type (cf. [2,6]). Finally, it would be interesting at this stage to refer the reader to the work of Dubinskiĭ [8], dealing with elliptic equations for a general class of operators of infinite order. The author proved, under hypothesis $(A_1$–$A_4)$ (see Sect. 3), the existence of solutions for the Dirichlet problem associated with the equation $Au = f$ in some functional Sobolev spaces of infinite order. Further, let us point out that in a recent work [4], the authors have studied some elliptic equations related to a class of operators of finite and infinite order and they proved the existence of solutions in the setting of anisotropic spaces.

The main point in our study is to consider separately some class of problems of kind (1) of finite and infinite order, and prove existence results in the setting of weighted anisotropic Sobolev spaces. Note that in the infinite order case, no monotonicity condition on the operator is imposed.

2 Preliminaries

Let Ω be a smooth bounded domain in $\mathbb{R}^N$. Almost everywhere positive and locally integrable function $\omega : \Omega \longrightarrow \mathbb{R}$ will be called a weight. We shall denote by $L^p_\omega(\Omega)$ the set of all measurable functions u on Ω such that the norm

$$\|u\|_{L^p_\omega} \equiv \|u\|_{p,\omega} = \left(\int_\Omega |u|^p \omega(x)\, dx \right)^{\frac{1}{p}} \quad 1 \leq p < \infty$$

is finite. $L^p_\omega(\Omega)$ is also called weighted Lebesgue space.

Further $p_\alpha \geq 1$ are real numbers and ω_α are weight measurable functions for all multi-indices α. For a positive integer m, we define the following vectors

$$\overrightarrow{p} = \{p_\alpha, |\alpha| \leq m\} \quad \text{and} \quad \overrightarrow{\omega} = \{\omega_\alpha = \omega_\alpha(x), |\alpha| \leq m\}.$$

The weighted anisotropic Sobolev space $W^{m,\overrightarrow{p}}(\Omega, \overrightarrow{\omega})$ is defined as the collection of all functions $u \in L^1_{loc}(\Omega)$, having the generalized derivatives $D^\alpha u$, $|\alpha| \leq m$ with the finite norm

$$\|u\| = \sum_{|\alpha|=0}^{m} \|D^\alpha u\|_{p_\alpha, \omega_\alpha}. \tag{2}$$

We denote by $C_0^\infty(\Omega)$ the space of all functions with compact support in Ω with continuous derivatives of arbitrary order.

We define the functional space $W_0^{m,\overrightarrow{p}}(\Omega, \overrightarrow{\omega})$ as the closure of $C_0^\infty(\Omega)$ in $W^{m,\overrightarrow{p}}(\Omega, \overrightarrow{\omega})$ with respect to the norm (2). Note that $C_0^\infty(\Omega)$ is dense in $W_0^{m,\overrightarrow{p}}(\Omega, \overrightarrow{\omega})$. By an adapted method of that of Adams [1], and by constructing an isometric isomorphism from $W^{m,\overrightarrow{p}}(\Omega, \overrightarrow{\omega})$ into $\prod_{|\alpha|=0}^{m} L^{p_\alpha}_{\omega_\alpha}(\Omega)$, we can show that $W_0^{m,\overrightarrow{p}}(\Omega, \omega)$ is separable and reflexive if $1 \leq p_\alpha < \infty$ and $1 < p_\alpha < \infty$, respectively, for all $|\alpha| \leq m$. For $p_\alpha > 1$, $|\alpha| \leq m$, $W^{-m,\overrightarrow{p'}}(\Omega, \overrightarrow{\omega}')$ designs its dual where $\overrightarrow{p'}$ is the conjugate of $\overrightarrow{p}$, i.e., $p'_\alpha = \frac{p_\alpha}{p_\alpha - 1}$ and $\overrightarrow{\omega}' = \{\omega_\alpha{}' = \omega_\alpha{}^{1-p'_\alpha}, |\alpha| \leq m\}$.

Let $a_\alpha > 0$ be a sequence of real numbers for all multi-indices α. The Sobolev space of infinite order is the functional space defined by

$$W^\infty(a_\alpha, p_\alpha, \omega_\alpha)(\Omega) = \left\{ u \in C^\infty(\Omega) : \ \rho(u) = \sum_{|\alpha|=0}^{\infty} a_\alpha \|D^\alpha u\|^{p_\alpha}_{p_\alpha, \omega\alpha} < \infty \right\}.$$

Since we shall deal with the Dirichlet problem, we shall use the functional space $W_0^\infty(a_\alpha, p_\alpha, \omega_\alpha)(\Omega)$ defined by

$$W_0^\infty(a_\alpha, p_\alpha, \omega_\alpha)(\Omega) = \left\{ u \in C_0^\infty(\Omega) : \ \rho(u) = \sum_{|\alpha|=0}^{\infty} a_\alpha \|D^\alpha u\|^{p_\alpha}_{p_\alpha, \omega_\alpha} < \infty \right\}.$$

We say that $W_0^{\infty}(a_\alpha, p_\alpha, \omega_\alpha)(\Omega)$ is a nontrivial space if it contains at least a nonzero function. The dual space of $W_0^{\infty}(a_\alpha, p_\alpha, \omega_\alpha)(\Omega)$ is defined as follows

$$W^{-\infty}(a_\alpha, p'_\alpha, \omega'_\alpha)(\Omega) = \Bigg\{ h \in D'(\Omega) : h = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha D^\alpha h_\alpha,\ \rho^{'}(h) = \sum_{|\alpha|=0}^{\infty} a_\alpha \|h_\alpha\|_{p'_\alpha,\omega'_\alpha}^{p'_\alpha} < \infty \Bigg\},$$

where $h_\alpha \in L_{\omega'_\alpha}^{p'_\alpha}(\Omega)$ and p'_α is the conjugate of p_α, i.e., $p'_\alpha = \frac{p_\alpha}{p_\alpha - 1}$ (for more details about classical weighted Sobolev spaces, we refer the reader to [7] and for non-weighted anisotropic Sobolev spaces of finite and infinite order, see [8,9]).

Let s be a real positive number, we denote by $E(s)$ the integer part of s and denote $\underline{p} = \min\{p_\alpha, |\alpha| \leq m\}$. We need the anisotropic Sobolev embeddings result.

Lemma 1 *Let Ω be a smooth bounded open subset of $\mathbb{R}^N$, and suppose that* $\inf \omega_\alpha > 0$ *a.e. in Ω for all $|\alpha| \leq m$.*

If $m\underline{p} < N$, then $W_0^{m,\overrightarrow{p}}(\Omega, \overrightarrow{\omega}) \subset L^q(\Omega)$ for all $q \in [\underline{p}, p^*[$ with $\frac{1}{p^*} = \frac{1}{\underline{p}} - \frac{m}{N}$.
If $m\underline{p} = N$, then $W_0^{m,\overrightarrow{p}}(\Omega, \overrightarrow{\omega}) \subset L^q(\Omega)$ for all $q \in [\underline{p}, +\infty[$.
If $m\underline{p} > N$, then $W_0^{m,\overrightarrow{p}}(\Omega, \overrightarrow{\omega}) \subset L^\infty(\Omega) \cap C^k(\overline{\Omega})$ where $k = E(m - \frac{N}{\underline{p}})$.

Moreover, the embeddings are compacts.

The proof follows immediately by using the fact that

$$W_0^{m,\overrightarrow{p}}(\Omega, \overrightarrow{\omega}) \subset W_0^{m,\overrightarrow{p}}(\Omega) \subset W_0^{m,\underline{p}}(\Omega).$$

Remark 1 A note concerning the anisotropic spaces $W_0^{m,\overrightarrow{p}}(\Omega)$ and their embedding theorems, can be found in [4].

3 Main results

In this section we formulate and prove the main result of the paper.

3.1 Finite order case

Consider the following strongly nonlinear problem with Dirichlet conditions

$$Au + g(x, u) = f \quad \text{in } \Omega.$$

Here, the function $g : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ is measurable and $f \in W^{-m,\overrightarrow{p'}}(\Omega, \overrightarrow{\omega}')$. Note that to deal with the Dirichlet problem, we use the space $W_0^{m,\overrightarrow{p}}(\Omega, \overrightarrow{\omega})$.

In the following we apply the theory of pseudo-monotone operators.

Definition 1 [5] Let Y be a reflexive Banach space. A bounded mapping B from Y to Y^* is called *pseudo-monotone* if for any sequence $u_n \in Y$ with $u_n \rightharpoonup u$ weakly in Y and $\limsup_{n\longrightarrow\infty}\langle Bu_n, u_n - v\rangle \leq 0$, one has

$$\liminf_{n\longrightarrow\infty}\langle Bu_n, u_n - v\rangle \geq \langle Bu, u - v\rangle \quad \text{for all } v \in Y.$$

We start by stating the following assumptions

(H) $A : W_0^{m,\overrightarrow{p}}(\Omega, \overrightarrow{\omega}) \mapsto W^{-m,\overrightarrow{p'}}(\Omega, \overrightarrow{\omega}')$ is a bounded operator, pseudo-monotone and coercive, i.e.,

$$\lim_{\|u\|\to+\infty}\frac{\langle Au, u\rangle}{\|u\|} = +\infty,$$

with $p_\alpha > 1$ and $\inf \omega_\alpha > 0$ a.e. in Ω for all $|\alpha| \leq m$.

(G) $g : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ is a Carathéodory function satisfying

$$\sup_{|u|\leq s} |g(x, u)| \leq h_s(x),$$

for a.e. $x \in \Omega$, all $s > 0$ and some function $h_s \in L^1(\Omega)$. We assume also the "sign condition" $g(x, u)u \geq 0$, for a.e. $x \in \Omega$ and for all $u \in \mathbb{R}$.

For the nonlinear Dirichlet boundary value problem (1), we state our main result as follows.

Theorem 1 *Let $m \in \mathbb{N}^*$ with $m\underline{p} > N$. Assume* (H) *and* (G) *hold true. Then for all $f \in W^{-m,\overrightarrow{p'}}(\Omega, \overrightarrow{\omega}')$, there exists at least one solution $u \in W_0^{m,\overrightarrow{p}}(\Omega, \overrightarrow{\omega})$ such that*

$$\begin{cases} g(x, u) \in L^1(\Omega), \ g(x, u)u \in L^1(\Omega) \\ \langle Au, v\rangle + \int_\Omega g(x, u)v\,dx = \langle f, v\rangle, \quad \forall v \in W_0^{m,\overrightarrow{p}}(\Omega, \overrightarrow{\omega}). \end{cases}$$

Proof Let $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that $0 < \varphi(x) < 1$ and $\varphi(x) = 1$ for x close to 0. Set

$$g_n(x, u) = \varphi\left(\frac{x}{n}\right) T_n g(x, u)$$

for a.e. $x \in \Omega$ for all $u \in W_0^{m,\overrightarrow{p}}(\Omega, \overrightarrow{\omega})$, where T_n is the usual truncation given by

$$T_n\xi = \begin{cases} \xi & si|\xi| < n \\ \frac{n\xi}{|\xi|} & si|\xi| \geq n. \end{cases}$$

Thanks to this truncation and as in Webb [11], we prove that there exists a $u_n \in W_0^{m,\overrightarrow{p}}(\Omega, \overrightarrow{\omega})$, which is the solution of the problem

$$Au_n + g_n(x, u_n) = f,$$

or in its variational formulation,

$$\langle Au_n, v\rangle + \int_\Omega g_n(x, u_n)v\,dx = \langle f, v\rangle, \quad \forall v \in W_0^{m,\overrightarrow{p}}(\Omega, \overrightarrow{\omega}). \tag{3}$$

Choosing $v = u_n$ as test function in (3), then in view of (H), (G) and Hölder inequality, we get the estimates

$$\|u_n\| \le K \tag{4}$$

$$\int_\Omega g_n(x, u_n)u_n\,dx \le K. \tag{5}$$

Now, since A is a bounded operator, we get

$$\|Au_n\|_{-m,\overrightarrow{p}',\overrightarrow{\omega}'} \le K \tag{6}$$

where K is a positive constant.

The reflexivity of the space $W_0^{m,\overrightarrow{p}}(\Omega, \overrightarrow{\omega})$ ($p_\alpha > 1$ for all $|\alpha| \le m$), and the estimates (4), (5) and (6) implies $u_n \rightharpoonup u$ weakly in $W_0^{m,\overrightarrow{p}}(\Omega, \overrightarrow{\omega})$ and $Au_n \rightharpoonup \chi$ weakly in $W^{-m,\overrightarrow{p'}}(\Omega, \overrightarrow{\omega}')$. In view of the Fatou's lemma, we obtain thanks to (5)

$$\int_\Omega g(x, u)u\,dx \le \lim_{n\to+\infty} \int_\Omega g_n(x, u_n)u_n\,dx \le K,$$

which implies that $g(x, u)u \in L^1(\Omega)$.

Let now $\delta > 0$. since $|g(x, s)|\delta \le |g(x, s)s|$ and then $|g(x, s)| \le \delta^{-1}|g(x, s)s|$ for $|s| \ge \delta$, we have

$$|g_n(x, u_n)| \le \sup_{|t|\le\delta} |g(x, t)| + \delta^{-1}|g_n(x, u_n)u_n| \le h_\delta(x) + \delta^{-1}|g_n(x, u_n)u_n|.$$

Let E be a measurable subset of Ω and let $\varepsilon > 0$, we have

$$\int_E |g_n(x, u_n)|\,dx \le \int_E h_\delta(x)\,dx + \delta^{-1}K,$$

where K is the constant of (5) which is independent of n. For $|E|$ sufficiently small and $\delta = \frac{2K}{\varepsilon}$, we obtain $\int_E |g_n(x, u_n)|\, dx \le \varepsilon$. Using Vitali's theorem we get

$$g_n(x, u_n) \to g(x, u) \ \text{ in } \ L^1(\Omega).$$

Hence it follows that $g(x, u) \in L^1(\Omega)$.

Therefore, passing to the limit in (3) and using the fact that $W_0^{m,\overrightarrow{p}}(\Omega, \overrightarrow{\omega}) \subset L^\infty(\Omega)$ with $m\underline{p} > N$ (see Lemma 1), we obtain

$$\langle \chi, v\rangle + \int_\Omega g(x, u)v\, dx = \langle f, v\rangle, \quad \forall v \in W_0^{m,\overrightarrow{p}}(\Omega, \overrightarrow{\omega}). \tag{7}$$

Now, we show that $\chi = Au$. Indeed, since (7) holds true for $v = u$, then by choosing $v = u_n$ as test function in (3), we deduce by Fatou's lemma that

$$\limsup_{n\to+\infty}\langle Au_n, u_n\rangle \le \langle f, u\rangle - \int_\Omega g(x, u)u\, dx = \langle \chi, u\rangle$$

which gives

$$\limsup_{n\to+\infty}\langle Au_n, u_n\rangle \le \langle \chi, u\rangle.$$

Finally, Since A is pseudo-monotone, we get $\chi = Au$. Consequently,

$$\begin{cases} g(x, u) \in L^1(\Omega),\ g(x, u)u \in L^1(\Omega) \\ \langle Au, v\rangle + \displaystyle\int_\Omega g(x, u)v\, dx = \langle f, v\rangle \quad \text{for all } v \in W_0^{m,\overrightarrow{p}}(\Omega, \overrightarrow{\omega}). \end{cases}$$

This completes the proof. □

3.2 Infinite order case

We denote by λ_α the number of multi-indices γ such that $|\gamma| \le |\alpha|$ and $A_\alpha : \Omega \times \mathbb{R}^{\lambda_\alpha} \mapsto \mathbb{R}$ is a real function. Here and so on, $A_\alpha(x, D^\gamma u)$ will denote the expression $A_\alpha(x, \dots, D^\gamma u, \dots)$ with $|\gamma| \le |\alpha|$. Let A be an operator of infinite order defined by

$$A(u) = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} D^\alpha A_\alpha(x, D^\gamma u), \quad |\gamma| \le |\alpha|.$$

Let us now formulate the assumptions

(A_1) $A_\alpha(x, \xi_\gamma)$ is a Carathéodory function for all α, $|\gamma| \le |\alpha|$.

(A_2) For a.e. $x \in \Omega$, all $m \in \mathbb{N}^*$, all $\xi_\gamma, \eta_\alpha, |\gamma| \leq |\alpha|$ and some constant $c_0 > 0$, we assume that

$$\left| \sum_{|\alpha|=0}^{m} A_\alpha(x, \xi_\gamma)\eta_\alpha \right| \leq c_0 \sum_{|\alpha|=0}^{m} a_\alpha \omega_\alpha |\xi_\alpha|^{p_\alpha - 1} |\eta_\alpha|,$$

where $a_\alpha > 0$, $p_\alpha > 1$ are reals numbers for all multi-indices α, and for all bounded sequence $(p_\alpha)_\alpha$. Here, ω_α are weight functions satisfying $\inf \omega_\alpha > 0$ a.e. in Ω for all $|\alpha| \leq m$.

(A_3) There exist constants $c_1 > 0, c_2 \geq 0$ such that for all $m \in \mathbb{N}^*$, for all $\xi_\gamma, \xi_\alpha; |\gamma| \leq |\alpha|$, we have

$$\sum_{|\alpha|=0}^{m} A_\alpha(x, \xi_\gamma)\xi_\alpha \geq c_1 \sum_{|\alpha|=0}^{m} a_\alpha \omega_\alpha |\xi_\alpha|^{p_\alpha} - c_2.$$

(A_4) The space $W_0^\infty(a_\alpha, p_\alpha, \omega_\alpha)(\Omega)$ is nontrivial.

(G_1) The function $g : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ is of Carathéodory type such that, for all $\delta > 0$,

$$\sup_{|u| \leq \delta} |g(x, u)| \leq h_\delta(x) \in L^1(\Omega).$$

(G_2) We assume the "sign condition" $g(x, u)u \geq 0$, for a.e. $x \in \Omega$ and all $u \in \mathbb{R}$.

Theorem 2 *Let us assume the conditions* (A_1) − (A_4), (G_1) *and* (G_2). *Then for all* $f \in W^{-\infty}(a_\alpha, p'_\alpha, \omega'_\alpha)(\Omega)$, *there exists* $u \in W_0^\infty(a_\alpha, p_\alpha, \omega_\alpha)(\Omega)$ *such that*

$$\begin{cases} g(x, u) \in L^1(\Omega), \;\; g(x, u)u \in L^1(\Omega) \\ \langle Au, v \rangle + \displaystyle\int_\Omega g(x, u)v \, dx = \langle f, v \rangle, \quad \forall v \in W_0^\infty(a_\alpha, p_\alpha, \omega_\alpha)(\Omega). \end{cases}$$

Proof In order to get our result, we will deal with steps.

Step (1): The approximate problem.

Let $m \in \mathbb{N}^*$ sufficiently large. Define the operator of order $2m + 2$ by

$$A_{2m+2}(u) = \sum_{|\alpha|=m+1} (-1)^{m+1} c_\alpha D^{2\alpha} u + \sum_{|\alpha|=0}^{m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, D^\gamma u), \quad |\gamma| \leq m.$$

Note that in the case of such an operator A_{2m+2} without weight, it was shown by Dubinskiĭ that c_α are constants small enough such that they fulfil the conditions of the following lemma introduced in [9].

Lemma 2 [9] *For all nontrivial space* $W_0^\infty(a_\alpha, p_\alpha)(\Omega)$, *there exists a nontrivial space* $W_0^\infty(c_\alpha, 2)(\Omega)$ *such that* $W_0^\infty(a_\alpha, p_\alpha)(\Omega) \subset W_0^\infty(c_\alpha, 2)(\Omega)$.

In our case, since $\inf \omega_\alpha > 0$ a.e. in Ω for all $|\alpha| \le m$, then $W_0^\infty(a_\alpha, p_\alpha, \omega_\alpha)(\Omega)$ is included in a space of infinite order "W" introduced by Dubinskiĭ [9] and which guarantees the existence of a nontrivial space $W_0^\infty(c_\alpha, 2)(\Omega)$ such that $W_0^\infty(a_\alpha, p_\alpha, \omega_\alpha)(\Omega) \subset W \subset W_0^\infty(c_\alpha, 2)(\Omega)$.

Now, the operator A_{2m+2} is clearly monotone since the term of higher order of derivation is linear and satisfies the monotonicity condition, this follows from the result of [10]. Moreover from assumptions (A_1), (A_2) and (A_3), we deduce that A_{2m+2} satisfies the growth, the coerciveness and the monotonicity conditions. Hence by Theorem 1, there exists an approximate solution u_m of the following problem

$$(\mathrm{Pb_m}) \begin{cases} g(x, u_m) \in L^1(\Omega), \quad g(x, u_m)u_m \in L^1(\Omega) \\ \langle A_{2m+2}(u_m), v\rangle + \int_\Omega g(x, u_m)v\,dx = \langle f_m, v\rangle, \quad \forall v \in W_0^{m+1,\overrightarrow{p}}(\Omega, \overrightarrow{\omega}). \end{cases}$$

with

$$f_m = \sum_{|\alpha|=0}^{m} (-1)^{|\alpha|} a_\alpha D^\alpha f_\alpha, \quad f_\alpha \in L_{\omega'_\alpha}^{p'_\alpha}(\Omega).$$

Here the derivatives are intended in distributional sense.

Step (2): A priori estimate.

Set $v = u_m$. Then in view of Young inequality, we have

$$\begin{aligned} \langle f_m, u_m\rangle &= \sum_{|\alpha|=0}^{m} \int_\Omega a_\alpha f_\alpha D^\alpha u_m\,dx \\ &= \sum_{|\alpha|=0}^{m} a_\alpha \int_\Omega \omega_\alpha^{-\frac{1}{p_\alpha}} f_\alpha \omega_\alpha^{\frac{1}{p_\alpha}} D^\alpha u_m\,dx \\ &\le \sum_{|\alpha|=0}^{m} \frac{a_\alpha}{p'_\alpha} \int_\Omega |f_\alpha|^{p'_\alpha} \omega_\alpha^{-\frac{p'_\alpha}{p_\alpha}}\,dx + \sum_{|\alpha|=0}^{m} \frac{a_\alpha}{p_\alpha} \int_\Omega |D^\alpha u_m|^{p_\alpha} \omega_\alpha\,dx \\ &\le \sum_{|\alpha|=0}^{\infty} a_\alpha \int_\Omega |f_\alpha|^{p'_\alpha} \omega_\alpha^{1-p'_\alpha}\,dx + \sum_{|\alpha|=0}^{m} \frac{a_\alpha}{p_\alpha} \int_\Omega |D^\alpha u_m|^{p_\alpha} \omega_\alpha\,dx. \end{aligned}$$

Now, using (A_3) and (G_2), we deduce the estimates

$$\sum_{|\alpha|=m+1} c_\alpha \|D^\alpha u_m\|_2^2 + \sum_{|\alpha|=0}^{m} (a_\alpha - \frac{a_\alpha}{p_\alpha}) \|D^\alpha u_m\|_{p_\alpha,\omega_\alpha}^{p_\alpha} \le \sum_{|\alpha|=0}^{\infty} a_\alpha \|f_\alpha\|_{p'_\alpha,\omega'_\alpha}^{p'_\alpha} \le K \quad (8)$$

and

$$\int_{\Omega} g(x, u_m)u_m\,dx \leq K \tag{9}$$

for some constant $K = K(f) > 0$. The estimate (8) is equivalent to

$$\sum_{|\alpha|=0}^{m+1} b_\alpha \|D^\alpha u_m\|_{p_\alpha,\omega_\alpha}^{p_\alpha} \leq K \tag{10}$$

with $b_\alpha = c_\alpha$, $p_\alpha = 2$ and $\omega_\alpha = 1$ for $|\alpha| = m+1$ and $b_\alpha = a_\alpha - \frac{a_\alpha}{p_\alpha}$ for $|\alpha| \leq m$. Consequently, we have

$$\|u_m\|_{W^{m+1,\vec{p}}(\Omega,\vec{\omega})} \leq K. \tag{11}$$

Then, in a similar way as in [9] and [4], via a diagonalization process, there exists a subsequence still, denoted by u_m, which converges uniformly to an element $u \in C_0^\infty(\Omega)$, also for all derivatives there holds $D^\alpha u_m \to D^\alpha u$.

Step (3): Convergence of problem (Pb_m).
There exists a solution u_m of problem (Pb_m), $m = 1, 2, \ldots$.

Then by passing to the limit, we have

$$\lim_{m\to+\infty} \langle A_{2m+2}(u_m), v\rangle + \lim_{m\to+\infty} \int_{\Omega} g(x, u_m)v\,dx = \lim_{m\to+\infty} \langle f_m, v\rangle,$$

for $v \in W_0^\infty(a_\alpha, p_\alpha, \omega_\alpha)(\Omega)$. It is clear that

$$\lim_{m\to+\infty} \langle f_m, v\rangle = \langle f, v\rangle \quad \text{for all} \quad v \in W_0^\infty(a_\alpha, p_\alpha, \omega_\alpha)(\Omega).$$

Now, we shall prove that

$$\lim_{m\to+\infty} \langle A_{2m+2}(u_m), v\rangle = \langle Au, v\rangle, \quad \text{for all } v \in W_0^\infty(a_\alpha, p_\alpha, \omega_\alpha)(\Omega).$$

In fact, let m_0 be a fixed number sufficiently large ($m > m_0$) and let $v \in W_0^\infty(a_\alpha, p_\alpha, \omega_\alpha)(\Omega)$. Set $\langle A(u) - A_{2m+2}(u_m), v\rangle = I_1 + I_2 + I_3$, where

$$I_1 = \sum_{|\alpha|=0}^{m_0} \langle A_\alpha(x, D^\gamma u) - A_\alpha(x, D^\gamma u_m), D^\alpha v\rangle,$$

$$I_2 = \sum_{|\alpha|=m_0+1}^{\infty} \langle A_\alpha(x, D^\gamma u), D^\alpha v\rangle$$

and

$$I_3 = - \sum_{|\alpha|=m_0+1}^{m} \langle A_\alpha(x, D^\gamma u_m), D^\alpha v\rangle - \sum_{|\alpha|=m+1} c_\alpha \langle D^\alpha u, D^\alpha v\rangle,$$

or in another form,

$$I_3 = - \sum_{|\alpha|=m_0+1}^{m+1} \langle A_\alpha(x, D^\gamma u_m), D^\alpha v\rangle$$

with $A_\alpha(x, \xi_\gamma) = c_\alpha \xi_\alpha$ and $c_\alpha \geq 0$ for $|\alpha| = m + 1$.

The aim is to prove that I_1, I_2 and I_3 tend to 0. On the one hand, since $A_\alpha(x, \xi_\gamma)$ is of Carathéodory type, $I_1 \to 0$, and the term I_2 is the remainder of a convergent series, hence $I_2 \to 0$. On the other hand, in view of (A_2) and Hölder inequality we have

$$\begin{aligned}
\left|\sum_{|\alpha|=m_0+1}^{m+1} \langle A_\alpha(x, D^\gamma u_m), D^\alpha v\rangle\right| &\leq \sum_{|\alpha|=m_0+1}^{m+1} |\langle A_\alpha(x, D^\gamma u_m), D^\alpha v\rangle| \\
&\leq c_0 \sum_{|\alpha|=m_0+1}^{m+1} a_\alpha \int_\Omega \omega_\alpha |D^\alpha u_m|^{p_\alpha - 1} |D^\alpha v|\, dx \\
&= c_0 \sum_{|\alpha|=m_0+1}^{m+1} a_\alpha \int_\Omega \omega_\alpha^{\frac{1}{p'_\alpha}} |D^\alpha u_m|^{p_\alpha - 1} \omega_\alpha^{\frac{1}{p_\alpha}} |D^\alpha v|\, dx \\
&\leq c_0 \sum_{|\alpha|=m_0+1}^{m+1} a_\alpha \|D^\alpha u_m\|_{p_\alpha, \omega_\alpha}^{p_\alpha - 1} \|D^\alpha v\|_{p_\alpha, \omega_\alpha}.
\end{aligned}$$

Therefore for all $\varepsilon > 0$, there holds $k(\varepsilon) > 0$ (see [5, p. 56]) such that

$$\begin{aligned}
\left|\sum_{|\alpha|=m_0+1}^{m+1} \langle A_\alpha(x, D^\gamma u_m), D^\alpha v\rangle\right| &\leq \varepsilon c_0 \sum_{|\alpha|=m_0+1}^{m+1} a_\alpha \|D^\alpha u_m\|_{p_\alpha, \omega_\alpha}^{p_\alpha} \\
&\quad + c_0 k(\varepsilon) \sum_{|\alpha|=m_0+1}^{m+1} a_\alpha \|D^\alpha v\|_{p_\alpha, \omega_\alpha}^{p_\alpha} \\
&\leq \varepsilon c_0 K + c_0 k(\varepsilon) \sum_{|\alpha|=m_0+1}^{\infty} a_\alpha \|D^\alpha v\|_{p_\alpha, \omega_\alpha}^{p_\alpha},
\end{aligned}$$

where K is the constant given in the estimate (8). Since the sequence (p_α) is bounded and $\sum_{|\alpha|=m_0+1}^{\infty} a_\alpha \|D^\alpha v\|_{p_\alpha,\omega_\alpha}^{p_\alpha}$ is the remainder of a convergent series, therefore $I_3 \to 0$ holds. Hence

$$\langle A_{2m+2}(u_m), v\rangle \to \langle A(u), v\rangle \quad \text{as} \quad m \to +\infty$$

for all $v \in W_0^\infty(a_\alpha, p_\alpha, \omega_\alpha)(\Omega)$.

Now we prove that

$$\lim_{m\to+\infty} \int_\Omega g(x, u_m)v\, dx = \int_\Omega g(x, u)v\, dx.$$

Indeed, we have $u_m \to u$ uniformly in Ω, hence $g(x, u_m) \to g(x, u)$ for a.e. $x \in \Omega$. In view of the Fatou's lemma and (9), we obtain

$$\int_\Omega g(x, u)u\, dx \le \lim_{m\to+\infty} \int_\Omega g(x, u_m)u_m\, dx \le K.$$

This implies $g(x, u)u \in L^1(\Omega)$. On the other hand, let $\delta > 0$, since $|g(x, s)|\delta \le |g(x, s)s|$ and then $|g(x, s)| \le \delta^{-1}|g(x, s)s|$ for $|s| \ge \delta$, we have

$$|g(x, u_m)| \le \sup_{|t|\le\delta} |g(x, t)| + \delta^{-1}|g(x, u_m)u_m| \le h_\delta(x) + \delta^{-1}|g(x, u_m)u_m|.$$

If E is a measurable subset of Ω and $\varepsilon > 0$, we have

$$\int_E |g(x, u_m)|\, dx \le \int_E h_\delta(x)\, dx + \delta^{-1}K,$$

where K is the constant of (9) which is independent of m. For $|E|$ sufficiently small and $\delta = \frac{2K}{\varepsilon}$, we obtain $\int_E |g(x, u_m)|\, dx \le \varepsilon$. Using Vitali's theorem we get

$$g(x, u_m) \to g(x, u) \quad \text{in} \quad L^1(\Omega).$$

Hence it follows that $g(x, u) \in L^1(\Omega)$.

By passing to the limit, we obtain

$$\langle Au, v\rangle + \int_\Omega g(x, u)v\, dx = \langle f, v\rangle, \quad \text{for all } v \in W_0^\infty(a_\alpha, p_\alpha, \omega_\alpha)(\Omega).$$

Finally, we conclude that

$$\begin{cases} g(x,u) \in L^1(\Omega), \ \ g(x,u)u \in L^1(\Omega) \\ \langle Au, v\rangle + \int_\Omega g(x,u)v\,dx = \langle f, v\rangle, \quad \forall v \in W_0^\infty(a_\alpha, p_\alpha, \omega_\alpha)(\Omega). \end{cases}$$

This achieved the proof. □

Remark 2 1. When $\omega_\alpha = 1$ for all $|\alpha| \leq m$, we find the anisotropic case introduced by Benkirane et al. [4], where the authors study some problems of elliptic equations of finite and infinite order. Moreover if $p_\alpha = p$ for all $|\alpha| \leq m$, we get the result established by Brézis and Browder in [6] for $mp > N$.
2. Note that the existence result is given without assuming the segment propriety of Ω, nor growth conditions on the parameters p_α. Moreover, in the infinite order case, no monotonicity condition on the operator is imposed.

Example 1. Let Ω be a bounded open set in $\mathbb{R}^N$, $N \geq 2$, with Lipschitz boundary $\partial\Omega$. Let $m = 1$ and consider the Carathéodory functions

$$A_i(x,s,\xi) = \omega_i |\xi_i|^{p_i - 1}\mathrm{sgn}(\xi_i), \quad \text{for } i = 1,\ldots,N.$$

where $\omega_i(x)$ are a given weight functions strictly positive almost everywhere in Ω. We can easily see that $A_i(x,s,\xi)$ are Carathéodory functions satisfying the condition (H).
2. As model of operators satisfying the condition (H) of Theorem 1, we consider nonlinear operator of order $2m$ defined as

$$Au = \sum_{|\alpha|=0}^{m} (-1)^{|\alpha|} D^\alpha(A_\alpha(x, D^\gamma u)), \quad |\gamma| \leq |\alpha|$$

where λ_α is the number of multi-indices γ such that $|\gamma| \leq |\alpha|$ and where $A_\alpha : \Omega \times \mathbb{R}^{\lambda_\alpha} \mapsto \mathbb{R}$ is a real function satisfying the following assumptions
(A_1) $A_\alpha(x, \xi_\gamma)$ is a Carathéodory function for all α, $|\gamma| \leq |\alpha|$.
(A_2) For a.e. $x \in \Omega$, all ξ_γ, η_α, $|\gamma| \leq |\alpha|$ and some constant $c_0 > 0$, we assume that

$$\left| \sum_{|\alpha|=0}^{m} A_\alpha(x,\xi_\gamma)\eta_\alpha \right| \leq c_0 \sum_{|\alpha|=0}^{m} \omega_\alpha |\xi_\alpha|^{p_\alpha - 1}|\eta_\alpha|,$$

where $p_\alpha > 1$ are real numbers and ω_α are weight functions such that $\inf \omega_\alpha > 0$ for all multi-indices $|\alpha| \leq m$.
(A_3) There exist constants $c_1 > 0$, $c_2 \geq 0$ such that for all ξ_γ, ξ_α; $|\gamma| \leq |\alpha|$, we have

$$\sum_{|\alpha|=0}^{m} A_\alpha(x,\xi_\gamma)\xi_\alpha \geq c_1 \sum_{|\alpha|=0}^{m} \omega_\alpha |\xi_\alpha|^{p_\alpha} - c_2.$$

Moreover, we assume that the operator A satisfies the following monotonicity condition

$$\sum_{|\alpha|=0}^{m} (A_\alpha(x, \xi_\gamma) - A_\alpha(x, \xi_\gamma^*))(\xi_\alpha - \xi_\alpha^*) > 0$$

for almost all $x \in \Omega$ and for all $\xi, \xi^* \in \mathbb{R}^N$ with $\xi \neq \xi^*$.

3. Some ideas of the following example of an operator of infinite order are closely inspired from the one used in [8]. Let us consider the operator

$$Au = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} D^\alpha \left(a_\alpha \omega_\alpha |D^\alpha u|^{p_\alpha - 2} D^\alpha u \right),$$

$a_\alpha \geq 0$, $p_\alpha > 1$ are real numbers and ω_α are weight functions such that the space $W_0^\infty(a_\alpha, p_\alpha, \omega_\alpha)(\Omega)$ is nontrivial, then the conditions (A_1), (A_2) and (A_3) are satisfied.

4. As a simple example of a function g satisfying the conditions of Theorems 1 and 2, consider the following function defined by $g(x, s) = \operatorname{sgn}(s) \exp(s) h(x)$ where $h \in L^1(\Omega)$, $h(x) \geq 0$ a.e. We can easily verify that it satisfies the conditions (G_1) and (G_2).

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