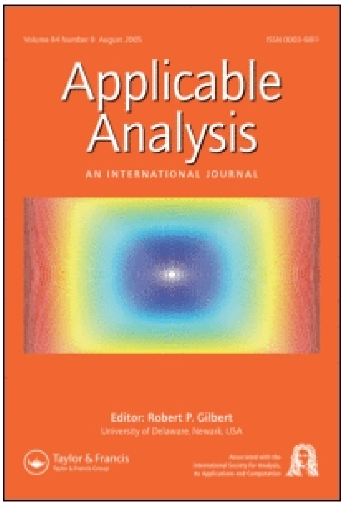

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On a strongly anisotropic equation with L^1 data

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In this article, we shall be concerned with the existence of solutions for the strongly non-linear boundary value problem:

$$Au + g(x, u) = f,$$

where A is an elliptic operator of finite order defined from an anisotropic Sobolev space of order m to its dual, g is a Carathéodory function satisfying essentially a sign condition on u with no growth restrictions and f belongs to L^1.

Keywords: strongly non-linear problem; anisotropic spaces; finite order; L^1 data; monotonicity condition; sign condition

Mathematics subject classifications: 35J60; 35J65; 35J70

1. Introduction

Let Ω be an open bounded subset of $\mathbb{R}^N$, $N \geq 2$. Consider the following strongly non-linear Dirichlet problem:

$$Au + g(x, u) = f \quad \text{in } \Omega, \tag{1.1}$$

where A is a non-linear elliptic operator satisfying certain monotonicity and coerciveness conditions. The non-linear term g has to fullfil only the sign condition $g(x, u)u \geq 0$, but we do not assume any growth conditions with respect to $|u|$ and f is a given data.

For a positive integer m, let $W^{m,\vec{p}}(\Omega)$ be the generalized Sobolev space associated to the vector $\vec{p}$, also called anisotropic space, with $\vec{p}$ denoting a vector of real numbers, i.e. $\vec{p} = \{p_\alpha, |\alpha| \leq m\}$, where p_α are real numbers for all multi-indices α.

In the variational case (i.e. where f belongs to the dual space), it is well known that the solvability of (1.1) is done by many authors and a lot of papers have appeared in the classical case when the operator A is of Leray–Lions type. In particular, Webb [1] has studied the isotropic case for the problem (1.1) and proved the existence of at least one solution u in the Sobolev space $W_0^{m,p}(\Omega)$ $(m \geq 1, 1 < p < \infty)$. In this context of Leray–Lions operators, some other existence results are given by Brezis and Browder in

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[2] in the classical Sobolev space and by Benkirane and Gossez [3] in the case of Sobolev–Orlicz space. Let us point out that in the classical case, the regularity of Ω is required (at least the segment property), since the techniques used are based on an Hedberg's type approximation (cf [2,3]). Finally, it would be interesting at this stage to refer the reader to the recent work [4], dealing with elliptic equations for a general class of operators of finite and infinite order. The authors proved the existence of solutions in anisotropic spaces.

In the L^1 case, if A is a Leray–Lions operator, let us recall that several studies have been devoted to the investigation of related problems and a lot of articles have appeared (cf [5–8]). Other problems have been considered in this direction, see e.g. [11–12]. It is our purpose in this article to solve the strongly non-linear problem associated to (1.1) in the case where $f \in L^1(\Omega)$ by assuming neither regularity condition on Ω nor restrictions on the parameters p_α.

2. Preliminaries

Let Ω be a bounded domain in $\mathbb{R}^N$. Further $p_\alpha \geq 1$ are real numbers for all multi-indices α, and $\|\cdot\|_{p_\alpha}$ is the usual norm in the Lebesgue space $L^{p_\alpha}(\Omega)$. For a positive integer m, we define the following vector of real numbers:

$$\vec{p} = \{p_\alpha, |\alpha| \leq m\},$$

and denote $\underline{p} = \min\{p_\alpha, |\alpha| \leq m\}$.

Now, let us consider the generalized functional Sobolev space

$$W^{m,\vec{p}}(\Omega) = \{u \in L^{p_0}(\Omega), D^\alpha u \in L^{p_\alpha}(\Omega), 0 < |\alpha| \leq m\}$$

equipped with the norm

$$\|u\| = \sum_{|\alpha|=0}^{m} \|D^\alpha u\|_{p_\alpha}. \tag{2.1}$$

We denote by $C_0^\infty(\Omega)$ the space of all functions with compact support in Ω with continuous derivatives of arbitrary order.

Since we shall deal with the Dirichlet problem, we shall define the functional space $W_0^{m,\vec{p}}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{m,\vec{p}}(\Omega)$ with respect to the norm (2.1). Note that $C_0^\infty(\Omega)$ is dense in $W_0^{m,\vec{p}}(\Omega)$. Both of $W^{m,\vec{p}}(\Omega)$ and $W_0^{m,\vec{p}}(\Omega)$ are separable if $1 \leq p_\alpha < \infty$ and reflexive if $1 < p_\alpha < \infty$ for all $|\alpha| \leq m$ (the proof of this is an adaptation from Adams [9]). For $p_\alpha > 1$, $W^{-m,\vec{p}'}(\Omega)$ designs its dual where $\vec{p}'$ is the conjugate of $\vec{p}$, i.e, $p'_\alpha = p_\alpha/(p_\alpha - 1)$ for all $|\alpha| \leq m$.

Remark 2.1 The techniques used in [9] allow us to adapt it to construct an isomorphism, so that we can easily show that the dual $W^{-m,\vec{p}'}(\Omega)$, can be identified to the set:

$$\Im_{m,\vec{p}} = \{T \in D'(\Omega)/\exists v \in L^{\vec{p}'}(\Omega) : T = \sum_{0 \leq |\alpha| \leq m} (-1)^{|\alpha|} D^\alpha v_\alpha\},$$

with

$$L^{\vec{p}'}(\Omega) = \prod_{0 \leq |\alpha| \leq m} L^{p'_\alpha}(\Omega).$$

This kind of anisotropic spaces and other of infinite order have been adapted by the authors for the study of some problems of elliptic equations. For more details, the reader can refer to [4].

We need the anisotropic Sobolev embeddings result.

LEMMA 2.1 *Let Ω be a bounded open subset of $\mathbb{R}^N$.*

If $m \cdot \underline{p} < N$, then $W_0^{m,\vec{p}}(\Omega) \subset L^q(\Omega)$ for all $q \in [\underline{p}, p^*[$ with $(1/p^*) = (1/\underline{p}) - (m/N)$.
If $m \cdot \underline{p} = N$, then $W_0^{m,\vec{p}}(\Omega) \subset L^q(\Omega)$ for all $q \in [\underline{p}, +\infty[$.
If $m \cdot \underline{p} > N$, then $W_0^{m,\vec{p}}(\Omega) \subset L^\infty(\Omega) \cap C^k(\overline{\Omega})$ where $k = E(m - (N/\underline{p}))$.

Moreover, the embeddings are compacts.

The proof follows immediately from the corresponding embedding theorems in the isotropic case by using the fact that $W^{m,\vec{p}}(\Omega) \subset W^{m,\underline{p}}(\Omega)$.

3. Main results

In this section, we formulate and prove the main result of the article. Consider the following strongly non-linear problem with Dirichlet conditions:

$$Au + g(x,u) = f \quad \text{in } \Omega.$$

Here, the function $g: \Omega \times \mathbb{R} \mapsto \mathbb{R}$ is measurable and $f \in L^1(\Omega)$. Note that to deal with the Dirichlet problem, we use the space $W_0^{m,\vec{p}}(\Omega)$.

In the following, we apply the theory of pseudo-monotone operators.

Definition 3.1 [10] Let Y be a reflexive Banach space. A bounded mapping B from Y to Y^* is called pseudo-monotone if for any sequence $u_n \in Y$ with $u_n \rightharpoonup u$ weakly in Y and $\limsup_{n\to\infty}\langle Bu_n, u_n - v\rangle \leq 0$, one has

$$\liminf_{n\longrightarrow\infty}\langle Bu_n, u_n - v\rangle \geq \langle Bu, u - v\rangle \quad \text{for all } v \in Y.$$

We start by stating the following assumptions:

(H) $A: W_0^{m,\vec{p}}(\Omega) \longmapsto W^{-m,\vec{p}'}(\Omega)$ is a bounded operator, pseudo-monotone and coercive, i.e.

$$\lim_{\|u\|_{m,\vec{p}}\to+\infty} \frac{\langle Au, u\rangle}{\|u\|_{m,\vec{p}}} = +\infty,$$

where $\vec{p} = \{p_\alpha, |\alpha| \leq m\}$ with $p_\alpha > 1$.

(G) $g: \Omega \times \mathbb{R} \mapsto \mathbb{R}$ is a Carathéodory function satisfying

$$\sup_{|u|<s} |g(x,u)| \leq h_s(x),$$

for a.e. $x \in \Omega$, all $s > 0$ and some function $h_s \in L^1(\Omega)$. We assume also the 'sign condition' $g(x,u)u \geq 0$, for a.e. $x \in \Omega$ and for all $u \in \mathbb{R}$.

For the non-linear Dirichlet boundary value problem (1.1), we state our main result as follows.

THEOREM 3.1 *Let $m \in \mathbb{N}^*$ such that $m\underline{p} > N$. Assume* (H) *and* (G) *hold true. Then for all* $f \in L^1(\Omega)$, *there exists at least one solution* $u \in W_0^{m,\vec{p}}(\Omega)$ *such that*

$$\begin{cases} g(x,u) \in L^1(\Omega), & g(x,u)u \in L^1(\Omega) \\ \langle Au, v\rangle + \displaystyle\int_\Omega g(x,u)v\,\mathrm{d}x = \langle f, v\rangle, & \forall v \in W_0^{m,\vec{p}}(\Omega). \end{cases}$$

Proof In order to prove our result, we proceed by steps.

Step 1 The approximate problem.

Let $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that $0 \le \varphi(x) \le 1$ and $\varphi(x) = 1$ for x close to 0. Set

$$f_n(x) = \varphi\left(\frac{x}{n}\right) P_n f(x)$$

with

$$P_n\xi = \begin{cases} \xi & \text{if } |\xi| < n \\ \dfrac{n\xi}{|\xi|} & \text{if } |\xi| \ge n. \end{cases}$$

It is clear that $|f_n| \le n$ for a.e. $x \in \Omega$. Thus, it follows that $f_n \in L^\infty(\Omega)$. Using Lebesgue's dominated convergence theorem, since $f_n \to f$ a.e. $x \in \Omega$ and $|f_n| \le |f| \in L^1(\Omega)$, we conclude that $f_n \to f$ strongly in $L^1(\Omega)$. Thanks to [[4], theorem 3.2], we deduce that for all $f_n \in L^\infty(\Omega)$, there exists at least one solution $u_n \in W_0^{m,\vec{p}}(\Omega)$ of the following approximate problem:

$$(\mathrm{P}_n)\begin{cases} g(x,u_n) \in L^1(\Omega), \quad g(x,u_n)u_n \in L^1(\Omega) \\ \langle A(u_n), v\rangle + \displaystyle\int_\Omega g(x,u_n)v\,\mathrm{d}x = \langle f_n, v\rangle, \quad \forall v \in W_0^{m,\vec{p}}(\Omega) \end{cases}$$

Step 2 *A priori* estimates.

By choosing $v = u_n$ as test function in (P_n) and in view of (H), (G), the Hölder inequality and by using the fact that $|f_n| \le |f|$, we get the estimates:

$$\|u_n\| \le K \tag{3.1}$$

$$\int_\Omega g(x,u_n)u_n\,\mathrm{d}x \le K \tag{3.2}$$

for some constant $K > 0$. Now, since A is a bounded operator, we get

$$\|Au_n\|_{-m,\vec{p}'} \le K. \tag{3.3}$$

Step 3 Convergence of u_n.

The reflexivity of the space $W_0^{m,\vec{p}}(\Omega)$ ($p_\alpha > 1$ for all $|\alpha| \le m$), and the estimates (3.1)–(3.3) implies $u_n \rightharpoonup u$ weakly in $W_0^{m,\vec{p}}(\Omega)$ and $Au_n \rightharpoonup \chi$ weakly in $W^{-m,\vec{p}'}(\Omega)$. In view of the Fatou's lemma, we obtain, thanks to (3.2),

$$\int_\Omega g(x,u)u\,\mathrm{d}x \le \lim_{n\to+\infty}\int_\Omega g(x,u_n)u_n\,\mathrm{d}x \le K$$

which implies that $g(x,u)u \in L^1(\Omega)$. Let now $\delta > 0$. We have

$$|g(x,u_n)| \leq \sup_{|t|<\delta} |g(x,t)| + \delta^{-1}|g(x,u_n)u_n|$$
$$\leq h_\delta(x) + \delta^{-1}|g(x,u_n)u_n|.$$

Let E be a measurable subset of Ω and let $\varepsilon > 0$, we have

$$\int_E |g(x,u_n)|\mathrm{d}x \leq \int_E h_\delta(x)\mathrm{d}x + \delta^{-1}K,$$

where K is the constant of (3.2) which is independent of n. For $|E|$ sufficiently small and $\delta = (2K/\varepsilon)$, we obtain $\int_E |g(x,u_n)|\mathrm{d}x \leq \varepsilon$. Using Vitali's theorem, we get $g(x,u_n) \to g(x,u)$ in $L^1(\Omega)$. Hence it follows that $g(x,u) \in L^1(\Omega)$.

Step 4 Convergence of the approximate problem (P_n).

Since f_n strongly converges to f in $L^1(\Omega)$ and using the fact that $W_0^{m,\vec{p}}(\Omega) \subset L^\infty(\Omega)$ with $m\underline{p} > N$ (Lemma 2.1), we obtain by passing to the limit in (P_n),

$$\langle \chi, v\rangle + \int_\Omega g(x,u)v\,\mathrm{d}x = \langle f, v\rangle, \quad \forall v \in W_0^{m,\vec{p}}(\Omega). \tag{3.4}$$

Now, we show that $\chi = Au$. Indeed, since (3.4) holds true for $v = u$, then by choosing $v = u_n$ as test function in (P_n), we deduce by Fatou's lemma that

$$\limsup_{n\to+\infty} \langle Au_n, u_n\rangle \leq \langle f, u\rangle - \int_\Omega g(x,u)u\,\mathrm{d}x = \langle \chi, u\rangle$$

which gives

$$\limsup_{n\to+\infty} \langle Au_n, u_n\rangle \leq \langle \chi, u\rangle.$$

Finally, Since A is pseudo-monotone, we get $\chi = Au$. Consequently,

$$\begin{cases} g(x,u) \in L^1(\Omega), \quad g(x,u)u \in L^1(\Omega) \\ \langle Au, v\rangle + \int_\Omega g(x,u)v\,\mathrm{d}x = \langle f, v\rangle \qquad \text{for all } v \in W_0^{m,\vec{p}}(\Omega). \end{cases}$$

This completes the proof. ■

Remarks

(1) Note that the existence result is given without assuming any restrictions on the parameter p_α.
(2) Our result is generalized to anisotropic case since we attribute variable values to the parameter p_α. If we assume $p_\alpha = p$ for all $|\alpha| \leq m$, we refind the isotropic case given in [2].
(3) In general, some methods of approximation need at least the segment property of the domain, here we assume no regularity condition of Ω.

Examples:

(1) It is easy to see that the operator defined by

$$Au = \sum_{|\alpha|=0}^{m} (-1)^{|\alpha|} D^\alpha(|D^\alpha u|^{p_\alpha - 2} D^\alpha u),$$

where $p_\alpha > 1$ are real numbers for $|\alpha| \le m$, satisfies the condition (H).

(2) For a more class of operators which satisfies the condition (H) of Theorem 3.1., we consider as model the following non-linear operator of order $2m$ defined as

$$Au = \sum_{|\alpha|=0}^{m} (-1)^{|\alpha|} D^\alpha (A_\alpha(x, D^\gamma u)), \quad |\gamma| \le |\alpha|$$

where $A_\alpha\colon \Omega \times \mathbb{R}^{\lambda_\alpha} \mapsto \mathbb{R}$, with λ_α is the number of multi-indices γ such that $|\gamma| \le |\alpha|$, is a real function satisfying the following assumptions:

(A_1) $A_\alpha(x, \xi_\gamma)$, is a Carathéodory function for all α, $|\gamma| \le |\alpha|$.

(A_2) For a.e. $x \in \Omega$, all ξ_γ, η_α, $|\gamma| \le |\alpha|$ and some constant $c_0 > 0$, we assume that

$$\left| \sum_{|\alpha|=0}^{m} A_\alpha(x, \xi_\gamma) \eta_\alpha \right| \le c_0 \sum_{|\alpha|=0}^{m} a_\alpha |\xi_\alpha|^{p_\alpha - 1} |\eta_\alpha|,$$

where $a_\alpha \ge 0$, $p_\alpha > 1$ are real numbers for all multi-indices α.

(A_3) There exist constants $c_1 > 0$, $c_2 \ge 0$ such that for all ξ_γ, ξ_α;$|\gamma| \le |\alpha|$, we have

$$\sum_{|\alpha|=0}^{m} A_\alpha(x, \xi_\gamma) \cdot \xi_\alpha \ge c_1 \sum_{|\alpha|=0}^{m} a_\alpha |\xi_\alpha|^{p_\alpha} - c_2.$$

Moreover, we assume that the operator A satisfies the following monotonicity condition:

$$\sum_{|\alpha|=0}^{m} (A_\alpha(x, \xi_\gamma) - A_\alpha(x, \xi^*_\gamma))(\xi_\alpha - \xi^*_\alpha) > 0$$

for almost all $x \in \Omega$ and for all ξ, $\xi^* \in \mathbb{R}^N$ with $\xi \ne \xi^*$.

(3) The idea of this example comes from [1]. Let g be a function defined by $g(x,s) = \mathrm{sgn}(s)\exp(s)\, h(x)$ where $h \in L^1(\Omega)$, $h(x) \ge 0$ a.e. We can easily verify that it satisfies the conditions stated in (G).

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