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On a class of nonlinear anisotropic parabolic problems

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We prove the existence of weak solutions for the strongly nonlinear parabolic problem

$$u_t + Au + g(x,t,u) + \gamma|u|^{p_0-2}u = f$$

in the anisotropic Sobolev space $L^{\boldsymbol{p}}(0,T;W_0^{1,\boldsymbol{p}}(\Omega))$, where the data f are assumed to be in the dual, and the nonlinear term $g(x,t,s)$ has growth and sign conditions on s.

Keywords: anisotropic Sobolev space; strongly nonlinear parabolic problems; weak solution

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1. Introduction

This paper is motivated by the study of the parabolic problem associated with the following equations:

$$\begin{aligned} u_t + Au + g(x,t,u) + \gamma|u|^{p_0-2}u = f \qquad & \text{in } Q_T = \Omega \times (0,T), \\ u(x,t) = 0 \qquad & \text{on } S_T = \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) \qquad & \text{in } \Omega, \end{aligned}$$

where $u_0 \in L^2(\Omega)$, $\gamma > 0$ and $f \in L^{\boldsymbol{p}'}(0,T;W^{-1,\boldsymbol{p}'}(\Omega))$.

Here A is an operator of Leray–Lions type acting from $L^{\boldsymbol{p}}(0,T;W_0^{1,\boldsymbol{p}}(\Omega))$ into its dual $L^{\boldsymbol{p}'}(0,T;W^{-1,\boldsymbol{p}'}(\Omega))$ and defined by

$$Au=-\sum_{i=1}^{N}D^i a_i(x,t,\nabla u),$$

while $(a_i(x,t,\xi))_{i=1,\dots,N}$ are Carathéodory functions.

In the isotropic case with $p>2$, Boccardo *et al.* [10] studied the problem

$$\left.\begin{aligned} u_t'-\operatorname{div}(a(x,t,u,\nabla u))=\mu &\quad \text{in } Q_T=\Omega\times(0,T),\\ u=0 &\quad \text{on } S_T=\partial\Omega\times(0,T),\\ u(x,0)=0 &\quad \text{in } \Omega, \end{aligned}\right\} \tag{1.1}$$

where a is a Carathéodory function satisfying certain growth, coerciveness and monotonicity conditions and μ belongs to $\mathcal{M}(Q_T)$, the space of bounded Borel measure on Q_T. Boccardo *et al.* proved the existence of solutions for problem (1.1) and showed some existence and regularity results in specific cases.

In [8], Boccardo *et al.* considered the following nonlinear parabolic equation:

$$\left.\begin{aligned} u_t'+Au+\alpha_0|u|^{s-1}u=f &\quad \text{in } Q_T=\Omega\times(0,T),\\ u=0 &\quad \text{on } S_T=\partial\Omega\times(0,T),\\ u(x,0)=0 &\quad \text{in } \Omega, \end{aligned}\right\} \tag{1.2}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $T>0$, $\alpha_0>0$ and $s>(p(N+l)-N)/N$. A is the so-called p-Laplacian operator, $A(v)=-\operatorname{div}(|\nabla v|^{p-2}\nabla v)$, with $p>1+N/(N+1)$. They established the regularity of solutions for problem (1.2) when the data f are assumed to be merely in $L^1(\Omega)$.

In the case of anisotropic spaces $W^{1,\boldsymbol{p}}(\Omega)$, Bendahmane *et al.* studied in [7] (see also [9]) an approximation result to prove the existence of weak solutions for some strongly nonlinear boundary-value problems of the form

$$Au+g(x,u)=f \quad \text{in } \Omega.$$

In addition, we refer the reader to the interesting papers [2–5], which deal with anisotropic problems governed by a general class of anisotropic operators in both stationary and parabolic cases.

In our case if f belongs only in $L^1(Q_T)$ and $u_0\in L^1(\Omega)$, it does not make sense to talk about weak solutions. However, it is interesting to study the existence result of entropy solutions or renormalized solutions using the truncation functions.

Finally, in the framework of variable exponent Sobolev spaces, Bendahmane *et al.* [6] studied the following nonlinear parabolic problem:

$$\left.\begin{aligned} u_t'-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)=f &\quad \text{in } Q_T=\Omega\times(0,T),\\ u=0 &\quad \text{on } S_T=\partial\Omega\times(0,T),\\ u(x,0)=u_0(x) &\quad \text{in } \Omega, \end{aligned}\right\} \tag{1.3}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ and $f\in L^1(\Omega)$. They established the existence and uniqueness of renormalized solutions to (1.3). Moreover, they proved that if $p(\cdot) > 2 - 1/(N+1)$, then $u \in L^{q(\cdot)}(0,T;W_0^{1,q(\cdot)}(\Omega))$ for all continuous variable exponents $q(\cdot)$ on $\bar{\Omega}$ satisfying

$$1 \leqslant q(x) < \frac{N(p(x)-1)+p(x)}{N+1} \quad \text{for all } x \in \bar{\Omega}.$$

2. Preliminaries

In this section we list briefly some definitions and well-known facts about anisotropic Sobolev spaces and give a technical lemma.

Recall that anisotropic Sobolev spaces were introduced and studied by Nikolskiĭ [13], Slobodeckiĭ [14] and Troisi [15], and later by Trudinger [16] in the framework of Orlicz spaces. Let Ω be an open bounded domain in $\mathbb{R}^N$ with boundary $\partial\Omega$, and let $p_0, p_1, \ldots, p_N$ be $N+1$ real numbers with $1 < p_i < \infty$, $i = 0, 1, \ldots, N$. We define

$$\boldsymbol{p} = (p_0, p_1, \ldots, p_N), \quad \underline{p} = \min\{p_i,\ i = 0,1,2,\ldots,N\}, \quad D^0 u = u, \quad D^i u = \frac{\partial u}{\partial x_i},$$

for $i = 1, \ldots, N$. With a slight abuse of notation, we introduce the anisotropic Sobolev space,

$$W^{1,\boldsymbol{p}}(\Omega) = \{u \in L^{p_0}(\Omega) \text{ and } D^i u \in L^{p_i}(\Omega),\ i = 1,2,\ldots,N\},$$

endowed with the norm

$$\|u\|_{1,\boldsymbol{p}} = \sum_{i=0}^{N} \|D^i u\|_{L^{p_i}(\Omega)}. \tag{2.1}$$

We also define $W_0^{1,\boldsymbol{p}}(\Omega)$ as the closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{1,\boldsymbol{p}}(\Omega)$ with respect to the norm (2.1). The dual of $W_0^{1,\boldsymbol{p}}(\Omega)$ is denoted by $W^{-1,\boldsymbol{p}'}(\Omega)$, where

$$\boldsymbol{p}' = (p'_0, p'_1, \ldots, p'_N), \qquad \frac{1}{p'_i} + \frac{1}{p_i} = 1$$

(cf. [7]).

REMARK 2.1. Arguing as in [1], it can easily be seen that $W^{1,\boldsymbol{p}}(\Omega)$ is a separable Banach space, and reflexive if $1 < p_i < \infty$ for $i = 0, 1, \ldots, N$.

We shall use the following Sobolev embeddings later.

LEMMA 2.2. *Let Ω be a bounded open set in $\mathbb{R}^N$. Then the following embeddings are compact:*

(i) *if $\underline{p} < N$, then $W_0^{1,\boldsymbol{p}}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ for all $q \in [\underline{p}, \underline{p}^*[$, where*

$$\frac{1}{\underline{p}^*} = \frac{1}{\underline{p}} - \frac{1}{N};$$

(ii) *if $\underline{p} = N$, then $W_0^{1,\boldsymbol{p}}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ for all $q \in [\underline{p}, +\infty[$;*

(iii) *if $\underline{p} > N$, then $W_0^{1,\boldsymbol{p}}(\Omega) \hookrightarrow\hookrightarrow L^\infty(\Omega) \cap C^0(\bar{\Omega})$.*

The proof of this lemma follows from the fact that the embedding $W_0^{1,\boldsymbol{p}}(\Omega) \hookrightarrow W_0^{1,\underline{p}}(\Omega)$ is continuous, and in view of the classical embedding theorems of the Sobolev spaces.

Definition 2.3. We denote the dual of the Sobolev space $W_0^{1,\boldsymbol{p}}(\Omega)$ by $W^{-1,\boldsymbol{p}'}(\Omega)$, and for each $F \in W^{-1,\boldsymbol{p}'}(\Omega)$ there exists $f_i \in L^{p_i'}(\Omega)$ for $i = 0, \dots, N$, such that

$$F = f_0 - \sum_{i=1}^{N} D^i f_i.$$

Moreover, for all $u \in W_0^{1,\boldsymbol{p}}(\Omega)$ we have

$$\langle F, u \rangle = \sum_{i=0}^{N} \int_\Omega f_i D^i u \, \mathrm{d}x.$$

Furthermore, we define a norm on the dual space by

$$\|F\|_{-1,\boldsymbol{p}'} = \sum_{i=0}^{N} \|f_i\|_{p_i'}.$$

2.1. Parabolic spaces

Let $Q_T = \Omega \times (0,T)$ with $0 < T < \infty$. We introduce the parabolic space $L^{\boldsymbol{p}}(0,T; W^{1,\boldsymbol{p}}(\Omega))$ by

$$L^{\boldsymbol{p}}(0,T; W^{1,\boldsymbol{p}(x)}(\Omega)) = \left\{ u \text{ is measurable} \;\middle|\; \sum_{i=0}^{N} \int_0^T \|D^i u\|_{p_i}^{p_i} \, \mathrm{d}t < \infty \right\}, \tag{2.2}$$

equipped with the norm

$$\|u\|_{L^{\boldsymbol{p}}(0,T;W^{1,\boldsymbol{p}}(\Omega))} := \sum_{i=0}^{N} \|D^\alpha u\|_{L^{p_i}(Q_T)}.$$

We also introduce the functional space $L^{\boldsymbol{p}}(0,T; W_0^{1,\boldsymbol{p}}(\Omega))$ by

$$L^{\boldsymbol{p}}(0,T; W_0^{1,\boldsymbol{p}}(\Omega)) = \{ u \in L^{\boldsymbol{p}}(0,T; W^{1,\boldsymbol{p}}(\Omega)) \mid u = 0 \text{ on } \partial\Omega \times [0,T] \}. \tag{2.3}$$

The spaces $L^{\boldsymbol{p}}(0,T; W^{1,\boldsymbol{p}}(\Omega))$ and $L^{\boldsymbol{p}}(0,T; W_0^{1,\boldsymbol{p}}(\Omega))$ are separable and reflexive Banach spaces.

The dual space of $L^{\boldsymbol{p}}(0,T; W_0^{1,\boldsymbol{p}}(\Omega))$ is defined as follows:

$$L^{\boldsymbol{p}'}(0,T; W^{-1,\boldsymbol{p}'}(\Omega)) = \left\{ F = f_0 - \sum_{i=1}^{N} D^i f_i \text{ with } f_0 \in L^{p_0'}(Q_T),\ f_i \in L^{p_i'}(Q_T) \right\}, \tag{2.4}$$

and normed by

$$\|F\|_{L^{\boldsymbol{p}'}(0,T;W^{-1,\boldsymbol{p}'}(\Omega))} = \sum_{i=0}^{N} \|f_i\|_{L^{p_i'}(Q_T)}.$$

Finally, the duality of the spaces

$$L^{\boldsymbol{p}}(0,T;W_0^{1,\boldsymbol{p}}(\Omega)) \quad \text{and} \quad L^{\boldsymbol{p}'}(0,T;W^{-1,\boldsymbol{p}'}(\Omega))$$

is given by

$$\langle F, v\rangle = \sum_{i=0}^{N} \int_{Q_T} f_i D^i v(x)\,\mathrm{d}x \quad \text{for any } v \in L^{\boldsymbol{p}}(0,T;W_0^{1,\boldsymbol{p}}(\Omega)).$$

3. Essential assumptions

Let $Q_T = \Omega \times (0,T)$ with $0 < T < \infty$ and $2N/(N+2) < p_i < \infty$. We consider a Leray–Lions operator A acting from $L^{\boldsymbol{p}}(0,T;W_0^{1,\boldsymbol{p}}(\Omega))$ into its dual $L^{\boldsymbol{p}'}(0,T;W^{-1,\boldsymbol{p}'}(\Omega))$ defined by

$$Au = -\sum_{i=1}^{N} D^i a_i(x,t,\nabla u).$$

The functions $(a_i(x,t,\xi))_{i=1,\dots,N}$ are Carathéodory functions (measurable with respect to (x,t) in Q_T for every ξ in $\mathbb{R}^N$ and continuous with respect to ξ in $\mathbb{R}^N$ for almost every (x,t) in Q_T) satisfying

$$|a_i(x,t,\xi)| \leqslant \beta(K_i(x,t) + |\xi_i|^{p_i-1}) \quad \text{for } i = 1,\dots,N, \tag{3.1}$$

$$a_i(x,t,\xi)\xi_i \geqslant \alpha|\xi_i|^{p_i} \quad \text{for } i = 1,\dots,N, \tag{3.2}$$

and for all $\xi = (\xi_1,\dots,\xi_N)$ and $\xi' = (\xi'_1,\dots,\xi'_N)$ we have

$$(a_i(x,t,\xi) - a_i(x,t,\xi'))(\xi_i - \xi'_i) > 0 \quad \text{for } \xi_i \neq \xi'_i, \tag{3.3}$$

for almost every $(x,t) \in Q_T$ and all $\xi \in \mathbb{R}^N$, where $K_i(x,t)$ is a non-negative function lying in $L^{p'_i}(Q_T)$ and $\alpha, \beta > 0$.

The nonlinear operator g is a Carathéodory function that satisfies

$$|g(x,t,s)| \leqslant |s|^{p_0-1} + b(x,t), \tag{3.4}$$

$$g(x,t,s)s \geqslant 0, \tag{3.5}$$

for almost every $(x,t) \in Q_T$ and any $s \in \mathbb{R}$, with the function $b\colon \Omega \times (0,T) \to \mathbb{R}^+$ such that $b \in L^{p'_0}(Q_T)$.

3.1. Some technical lemmas

In this subsection, we state and prove the following technical lemmas that will be needed later.

Lemma 3.1 (Hewitt and Stromberg [11, theorem 13.47]). *Let $(u_n)_n$ be a sequence in $L^1(\Omega)$ and let $u \in L^1(\Omega)$ such that $u_n \to u$ almost everywhere (a.e.) in Ω, $u_n, u \geqslant 0$ a.e. and*

$$\int_\Omega u_n\,\mathrm{d}x \to \int_\Omega u\,\mathrm{d}x.$$

Then $u_n \to u$ in $L^1(\Omega)$.

LEMMA 3.2. *Let $g \in L^r(Q_T)$ and $g_n \in L^r(Q_T)$ with $\|g_n\|_{L^r(Q_T)} \leqslant C$ for $1 < r < \infty$. If $g_n(x,t) \to g(x,t)$ a.e. on Q_T, then $g_n \rightharpoonup g$ in $L^r(Q_T)$.*

LEMMA 3.3. *Assume that (3.1)–(3.3) hold, and let $(u_n)_n$ be a sequence in $L^{\boldsymbol{p}}(0,T; W_0^{1,\boldsymbol{p}}(\Omega))$ such that $u_n \rightharpoonup u$ in $L^{\boldsymbol{p}}(0,T;W_0^{1,\boldsymbol{p}}(\Omega))$ and*

$$\int_0^T \langle Au_n - Au, u_n - u\rangle \,\mathrm{d}t + \int_{Q_T} (|u_n|^{p_0-2}u_n - |u|^{p_0-2}u)(u_n-u)\,\mathrm{d}x\,\mathrm{d}t \to 0. \tag{3.6}$$

Then $u_n \to u$ in $L^{\boldsymbol{p}}(0,T;W_0^{1,\boldsymbol{p}}(Q_T))$ for a subsequence.

Proof. Let

$$D_n(x,t) = \sum_{i=1}^N (a_i(x,t,\nabla u_n) - a_i(x,t,\nabla u))(D^i u_n - D^i u) + (|u_n|^{p_0-2}u_n - |u|^{p_0-2}u)(u_n-u).$$

By (3.3), D_n is a positive function, and, in view of (3.6), $D_n \to 0$ in $L^1(Q_T)$ as $n \to \infty$.

We have $u_n \rightharpoonup u$ in $L^{\boldsymbol{p}}(0,T;W_0^{1,\boldsymbol{p}}(\Omega))$. Then $u_n \to u$ in $L^1(Q_T)$, and since $D_n \to 0$ a.e in Q_T there exists a subset B in Q_T with measure zero such that for all $(x,t) \in Q_T \setminus B$ we have

$$|u(x,t)| < \infty, \quad |\nabla u(x,t)| < \infty, \quad K_i(x,t) < \infty, \quad u_n \to u \quad \text{and} \quad D_n \to 0.$$

We also have

$$\begin{aligned}
D_n(x,t) &= \sum_{i=1}^N (a_i(x,t,\nabla u_n) - a_i(x,t,\nabla u))(D^i u_n - D^i u) \\
&\qquad + (|u_n|^{p_0-2}u_n - |u|^{p_0-2}u)(u_n-u) \\
&= \sum_{i=1}^N (a_i(x,t,\nabla u_n)D^i u_n + a_i(x,t,\nabla u)D^i u \\
&\qquad - a_i(x,t,\nabla u)D^i u_n - a_i(x,t,\nabla u_n)D^i u) \\
&\qquad + |u_n|^{p_0} + |u|^{p_0} - |u_n|^{p_0-2}u_n u - |u|^{p_0-2}u u_n \\
&\geqslant \underline{\alpha}\sum_{i=0}^N |D^i u_n|^{p_i} + \underline{\alpha}\sum_{i=0}^N |D^i u|^{p_i} - \beta\sum_{i=1}^N (K_i(x,t) + |D^i u|^{p_i-1})|D^i u_n| \\
&\qquad - \beta\sum_{i=1}^N (K_i(x,t) + |D^i u_n|^{p_i-1})|D^i u| - |u_n|^{p_0-1}|u| - |u|^{p_0-1}|u_n| \\
&\geqslant \underline{\alpha}\sum_{i=0}^N |D^i u_n|^{p_i} - C_{x,t}\sum_{i=0}^N (1 + |D^i u_n|^{p_i-1} + |D^i u_n|),
\end{aligned}$$

with $\underline{\alpha} = \min(\alpha, 1)$ and $C_{x,t}$ depending on x and t, without dependence on n. Hence, we obtain

$$D_n(x,t) \geqslant \sum_{i=0}^{N} |D^i u_n|^{p_i} \left(\underline{\alpha} - \frac{C_{x,t}}{|D^i u_n|^{p_i}} - \frac{C_{x,t}}{|D^i u_n|} - \frac{C_{x,t}}{|D^i u_n|^{p_i-1}} \right).$$

By standard arguments, $(D^i u_n)_n$ is bounded a.e. in Q_T for $i = 0, 1, \ldots, N$.[1]

Letting ξ_i^* be an accumulation point of $(D^i u_n)_n$ for $i = 1, \ldots, N$, we have $|\xi_i^*| < \infty$, and by the continuity of $a(x, t, \cdot)$ we obtain

$$[a_i(x,t,\xi^*) - a_i(x,t,\nabla u)](\xi_i^* - D^i u) = 0 \quad \text{for } i = 1, \ldots, N.$$

By (3.3), we have $\xi^* = \nabla u$. The uniqueness of the accumulation point implies that $\nabla u_n \to \nabla u$ a.e in Ω.

Now, since $(a_i(x,t,\nabla u_n))_n$ is bounded in $L^{p_i'}(Q_T)$ and

$$a_i(x,t,\nabla u_n) \to a_i(x,t,\nabla u) \quad \text{a.e in } Q_T$$

by lemma 3.2, we can establish that

$$a_i(x,t,\nabla u_n) \rightharpoonup a_i(x,t,\nabla u) \text{ in } L^{p_i'}(Q_T) \quad \text{for } i = 1, \ldots, N.$$

Using (3.6) and in view of lemma 3.1, we deduce that

$$|u_n|^{p_0} \to |u|^{p_0} \quad \text{in } L^1(Q_T). \tag{3.7}$$

Then $u_n \to u$ in $L^{p_0}(Q_T)$, and

$$a_i(x,t,\nabla u_n) D^i u_n \to a_i(x,t,\nabla u) D^i u \quad \text{in } L^1(Q_T). \tag{3.8}$$

According to the condition (3.2), we have

$$\alpha |D^i u_n|^{p_i} \leqslant a_i(x,t,\nabla u_n) D^i u_n \quad \text{for } i = 1, \ldots, N.$$

Letting

$$y_n^i = \frac{1}{\alpha} a_i(x,t,\nabla u_n) D^i u_n \quad \text{and} \quad y^i = \frac{1}{\alpha} a_i(x,t,\nabla u) D^i u,$$

in view of Fatou's lemma we get

$$\int_{Q_T} 2y^i \,\mathrm{d}x\,\mathrm{d}t \leqslant \liminf_{n\to\infty} \int_{Q_T} \left(y_n^i + y^i - \frac{1}{2^{p_i-1}} |D^i u_n - D^i u|^{p_i} \right) \mathrm{d}x\,\mathrm{d}t.$$

Hence,

$$0 \leqslant -\limsup_{n\to\infty} \int_{Q_T} |D^i u_n - D^i u|^{p_i} \,\mathrm{d}x\,\mathrm{d}t.$$

[1] Indeed, if $|D^i u_n| \to \infty$ in a measurable subset $E \subset Q_T$, then

$$\begin{aligned} &\lim_{n\to\infty} \int_{\Omega} D_n(x,t)\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad \geqslant \lim_{n\to\infty} \int_E |D^i u_n|^{p_i} \left(\underline{\alpha} - \frac{C_{x,t}}{|D^i u_n|^{p_i}} - \frac{C_{x,t}}{|D^i u_n|} - \frac{C_{x,t}}{|D^i u_n|^{p_i-1}} \right) \mathrm{d}x\,\mathrm{d}t \\ &\qquad = \infty, \end{aligned}$$

which is absurd since $D_n \to 0$ in $L^1(Q_T)$.

Thus, since

$$0 \leqslant \liminf_{n\to\infty} \int_{Q_T} |D^i u_n - D^i u|^{p_i} \,\mathrm{d}x\,\mathrm{d}t \leqslant \limsup_{n\to\infty} \int_{Q_T} |D^i u_n - D^i u|^{p_i} \,\mathrm{d}x\,\mathrm{d}t \leqslant 0,$$

it follows that

$$\int_{Q_T} |D^i u_n - D^i u|^{p_i} \,\mathrm{d}x\,\mathrm{d}t \to 0 \quad \text{as } n \to \infty.$$

Then, we obtain

$$D^i u_n \to D^i u \quad \text{in } L^{p_i}(Q_T) \quad \text{for } i = 1, \ldots, N.$$

Consequently, due to (3.7), we deduce that

$$u_n \to u \quad \text{in } L^{\boldsymbol{p}}(0,T; W_0^{1,\boldsymbol{p}}(Q_T)).$$

□

4. Main result

Now, we state the main result in our paper.

THEOREM 4.1. *Assume that (3.1)–(3.5) hold and let* $f \in L^{\boldsymbol{p}'}(0,T; W^{-1,\boldsymbol{p}'}(\Omega))$. *Then there exists at least one solution* $u \in L^{\boldsymbol{p}}(0,T; W_0^{1,\boldsymbol{p}}(\Omega))$ *of the problem*

$$\left.\begin{aligned} u_t + Au + g(x,t,u) + \gamma |u|^{p_0-2}u &= f && \text{in } Q_T = \Omega \times (0,T),\\ u(x,t) &= 0 && \text{on } S_T = \partial\Omega \times (0,T),\\ u(x,0) &= u_0(x) && \text{in } \Omega, \end{aligned}\right\} \tag{4.1}$$

where $u_0 \in L^2(\Omega)$.

Proof of theorem 4.1.

STEP 1 (approximate problem). Consider the following problem:

$$\left.\begin{aligned} \frac{\partial u_n}{\partial t} + Au_n + g_n(x,t,u_n) + \gamma |u|^{p_0-2}u &= f(x,t) && \text{in } \Omega \times (0,T),\\ u_k(x,t) &= 0 && \text{on } \partial\Omega \times (0,T),\\ u_k(x,0) &= u_0 && \text{in } \Omega, \end{aligned}\right\} \tag{4.2}$$

with

$$g_n(x,t,s) = \frac{g(x,t,s)}{1 + |g(x,t,s)|/n}.$$

Note that

$$g_n(x,t,s)s \geqslant 0, \quad |g_n(x,t,s)| \leqslant |g(x,t,s)| \quad \text{and} \quad |g_n(x,t,s)| \leqslant n \quad \forall n \in \mathbb{N}^*.$$

We define the operator $G_n : L^{\boldsymbol{p}}(0,T; W_0^{1,\boldsymbol{p}}(\Omega)) \to L^{\boldsymbol{p}'}(0,T; W^{-1,,\boldsymbol{p}'}(\Omega))$ by

$$\int_0^T \langle G_n u, v\rangle \,\mathrm{d}t = \int_{Q_T} g_n(x,t,u)v \,\mathrm{d}x\,\mathrm{d}t + \gamma \int_{Q_T} |u|^{p_0-2}uv \,\mathrm{d}x\,\mathrm{d}t \qquad \forall v \in L^{\boldsymbol{p}}(0,T; W_0^{1,\boldsymbol{p}}(\Omega)).$$

By the Hölder inequality, we have

$$\begin{aligned}\left|\int_0^T \langle G_n u, v\rangle \,\mathrm{d}t\right| &\leqslant \int_{Q_T} |g_n(x,t,u)||v| \,\mathrm{d}x\,\mathrm{d}t + \gamma \int_{Q_T} |u|^{p_0-1}|v| \,\mathrm{d}x\,\mathrm{d}t \\ &\leqslant \int_{Q_T} ((\gamma+1)|u|^{p_0-1} + b(x,t))|v| \,\mathrm{d}x\,\mathrm{d}t \\ &\leqslant ((\gamma+1)\|u\|^{p_0-1}_{L^{p_0}(Q_T)} + \|b(x,t)\|_{L^{p'_0}(Q_T)})\|v\|_{L^{p_0}(Q_T)} \end{aligned} \tag{4.3}$$

and

$$\begin{aligned}\left|\int_0^T \langle Au, v\rangle \,\mathrm{d}t\right| &\leqslant \sum_{i=1}^N \int_{Q_T} |a_i(x,t,\nabla u)||D^i v| \,\mathrm{d}x\,\mathrm{d}t \\ &\leqslant \sum_{i=1}^N \int_{Q_T} \beta(K_i(x,t) + |D^i u|^{p_i-1})|D^i v| \,\mathrm{d}x\,\mathrm{d}t \\ &\leqslant \beta \sum_{i=1}^N (\|K_i(x,t)\|_{L^{p'_i}(Q_T)} + \|D^i u\|^{p_i-1}_{L^{p_i}(Q_T)})\|D^i v\|_{L^{p_i}(Q_T)} \end{aligned} \tag{4.4}$$

for all $u, v \in L^{\boldsymbol{p}}(0,T;W_0^{1,\boldsymbol{p}}(\Omega))$.

Lemma 4.2. *The operator*

$$B_n = A + G_n$$

is pseudo-monotone from $L^{\boldsymbol{p}}(0,T;W_0^{1,\boldsymbol{p}}(\Omega))$ *into* $L^{\boldsymbol{p}'}(0,T;W^{-1,\boldsymbol{p}'}(\Omega))$. *Moreover,* B_n *is coercive in the following sense:*

$$\left(\frac{1}{\|v\|_{L^{\boldsymbol{p}}(0,T;W_0^{1,\boldsymbol{p}}(\Omega))}}\right)\int_0^T \langle B_n v, v\rangle \,\mathrm{d}t \to +\infty \quad \textit{as } \|v\|_{L^{\boldsymbol{p}}(0,T;W_0^{1,\boldsymbol{p}}(\Omega))} \to +\infty,$$

for $v \in L^{\boldsymbol{p}}(0,T;W_0^{1,\boldsymbol{p}}(\Omega))$.

Proof of lemma 4.2. In view of inequalities (4.3) and (4.4), the operator B_n is bounded. For the coercivity, due to (3.2) and (3.5), we have

$$\begin{aligned}\int_0^T \langle B_n u, u\rangle \,\mathrm{d}t &= \sum_{i=1}^N \int_{Q_T} a_i(x,t,\nabla u) D^i u \,\mathrm{d}x\,\mathrm{d}t \\ &\qquad + \int_{Q_T} g(x,t,u)u \,\mathrm{d}x\,\mathrm{d}t + \gamma \int_{Q_T} |u|^{p_0} \,\mathrm{d}x\,\mathrm{d}t \\ &\geqslant \alpha \sum_{i=1}^N \int_{Q_T} |D^i u|^{p_i} \,\mathrm{d}x\,\mathrm{d}t + \gamma \int_{Q_T} |u|^{p_0} \,\mathrm{d}x\,\mathrm{d}t \\ &\geqslant \alpha \sum_{i=1}^N (\|D^i u\|^{\underline{p}}_{L^{p_i}(Q_T)} - 1) + \gamma(\|u\|^{\underline{p}}_{L^{p_0}(Q_T)} - 1) \\ &\geqslant \alpha' \|v\|^{\underline{p}}_{L^{\boldsymbol{p}}(0,T;W_0^{1,\boldsymbol{p}}(\Omega))} - \alpha N - \gamma \end{aligned}$$

for all $u \in L^{\boldsymbol{p}}(0,T;W_0^{1,\boldsymbol{p}}(\Omega))$. It follows that

$$\left(\frac{1}{\|u\|_{L^{\boldsymbol{p}}(0,T;W_0^{1,\boldsymbol{p}}(\Omega))}}\right)\int_0^T \langle B_n u, u\rangle \,\mathrm{d}t \to +\infty \quad \text{as } \|u\|_{L^{\boldsymbol{p}}(0,T;W_0^{1,\boldsymbol{p}}(\Omega))} \to +\infty.$$

It remains to show that B_n is pseudo-monotone. Indeed, let $(u_k)_k$ be a sequence in $L^{\boldsymbol{p}}(0,T;W_0^{1,\boldsymbol{p}}(\Omega))$ such that

$$\left.\begin{aligned} u_k &\rightharpoonup u \quad \text{in } L^{\boldsymbol{p}}(0,T;W_0^{1,\boldsymbol{p}}(\Omega)),\\ B_n u_k &\rightharpoonup \chi \quad \text{in } L^{\boldsymbol{p}'}(0,T;W^{-1,\boldsymbol{p}'}(\Omega)),\\ \limsup_{k\to\infty}\int_0^T \langle B_n u_k, u_k\rangle \,\mathrm{d}t &\leqslant \int_0^T \langle \chi, u\rangle \,\mathrm{d}t. \end{aligned}\right\} \tag{4.5}$$

We shall prove that

$$\chi = B_n u \quad \text{and} \quad \int_0^T \langle B_n u_k, u_k\rangle \,\mathrm{d}t \to \int_0^T \langle \chi, u\rangle \,\mathrm{d}t \text{ as } k \to +\infty.$$

First, since the embedding $L^{\boldsymbol{p}}(0,T;W_0^{1,\boldsymbol{p}}(\Omega)) \hookrightarrow\hookrightarrow L^1(Q_T)$ is compact, there exists a subsequence, still denoted (u_k), such that $u_h \to u$ in $L^1(Q_T)$.

Second, we have that $(u_k)_k$ is a bounded sequence in $L^{\boldsymbol{p}}(0,T;W_0^{1,\boldsymbol{p}}(\Omega))$. Then, by (3.1), $(a_i(x,t,\nabla u_k))_k$ is bounded in $L^{p_i'}(Q_T)$. Therefore, there exists a function $\varphi_i \in L^{p_i'}(Q_T)$ such that

$$a_i(x,t,\nabla u_k) \rightharpoonup \varphi_i \text{ in } L^{p_i'}(Q_T) \quad \text{for } i = 1,\dots,N. \tag{4.6}$$

We have $g_n(x,t,u_k) \to g_n(x,t,u)$ a.e. in Q_T and $|g_n(x,t,u_k)| \leqslant n \in L^{p_0'}(Q_T)$. Then, by the Lebesgue dominated convergence theorem, we deduce that

$$g_n(x,t,u_k) \to g_n(x,t,u) \quad \text{in } L^{p_0'}(Q_T). \tag{4.7}$$

We also have

$$|u_k|^{p_0-2}u_k \rightharpoonup |u|^{p_0-2}u \quad \text{in } L^{p_0'}(Q_T). \tag{4.8}$$

Thus, for all $v \in L^{\boldsymbol{p}}(0,T;W_0^{1,\boldsymbol{p}}(\Omega))$, we get

$$\begin{aligned} \langle \chi, v\rangle &= \lim_{k\to\infty} \langle B_n u_k, v\rangle \\ &= \lim_{k\to\infty} \sum_{i=1}^N \int_{Q_T} a_i(x,t,\nabla u_k) D^i v \,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \lim_{k\to\infty} \int_{Q_T} g_n(x,t,u_k) v \,\mathrm{d}x\,\mathrm{d}t + \lim_{k\to\infty} \gamma \int_{Q_T} |u_k|^{p_0-2}u_k v \,\mathrm{d}x\,\mathrm{d}t \\ &= \sum_{i=1}^N \int_{Q_T} \varphi_i D^i v \,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} g_n(x,t,u) v \,\mathrm{d}x\,\mathrm{d}t + \gamma \int_{Q_T} |u|^{p_0-2} u v \,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{4.9}$$

By using (4.5) and (4.9), we obtain

$$\begin{aligned}\limsup_{k\to\infty}&\langle B_n(u_k),u_k\rangle\\ &=\limsup_{k\to\infty}\bigg\{\sum_{i=1}^N\int_{Q_T}a_i(x,t,\nabla u_k)D^iu_k\,\mathrm{d}x\,\mathrm{d}t\\ &\qquad\qquad+\int_{Q_T}g_n(x,t,u_k)u_k\,\mathrm{d}x\,\mathrm{d}t+\gamma\int_{Q_T}|u_k|^{p_0}\,\mathrm{d}x\,\mathrm{d}t\bigg\}\\ &\leqslant\sum_{i=1}^N\int_{Q_T}\varphi_iD^iu\,\mathrm{d}x\,\mathrm{d}t+\int_{Q_T}g_n(x,t,u)u\,\mathrm{d}x\,\mathrm{d}t+\gamma\int_{Q_T}|u|^{p_0}\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}\tag{4.10}$$

By (4.7), we have

$$\int_{Q_T}g_n(x,t,u_k)u_k\,\mathrm{d}x\,\mathrm{d}t\to\int_{Q_T}g_n(x,t,u)u\,\mathrm{d}x\,\mathrm{d}t.\tag{4.11}$$

Therefore,

$$\begin{aligned}\limsup_{k\to\infty}\bigg\{\sum_{i=1}^N\int_{Q_T}a_i(x,t,\nabla u_k)D^iu_k\,\mathrm{d}x\,\mathrm{d}t+\int_{Q_T}|u_k|^{p_0}\,\mathrm{d}x\,\mathrm{d}t\bigg\}&\\ \leqslant\sum_{i=1}^N\int_{Q_T}\varphi_iD^iu\,\mathrm{d}x\,\mathrm{d}t+\gamma\int_{Q_T}|u|^{p_0}\,\mathrm{d}x\,\mathrm{d}t.&\end{aligned}\tag{4.12}$$

On the other hand, by using (3.3) we get

$$\begin{aligned}\sum_{i=1}^N\int_{Q_T}(a_i(x,t,\nabla u_k)-a_i(x,t,\nabla u))(D^iu_k-D^iu)\,\mathrm{d}x\,\mathrm{d}t&\\ +\gamma\int_{Q_T}(|u_k|^{p_0-2}u_k-|u|^{p_0-2}u)(u_k-u)\,\mathrm{d}x\,\mathrm{d}t\geqslant 0.&\end{aligned}\tag{4.13}$$

Then, we obtain

$$\begin{aligned}\sum_{i=1}^N&\int_{Q_T}a_i(x,t,\nabla u_k)D^iu_k\,\mathrm{d}x\,\mathrm{d}t+\gamma\int_{Q_T}|u_k|^{p_0}\,\mathrm{d}x\,\mathrm{d}t\\ &\geqslant-\sum_{i=1}^N\int_{Q_T}a_i(x,t,\nabla u)D^iu\,\mathrm{d}x\,\mathrm{d}t-\gamma\int_{Q_T}|u|^{p_0}\,\mathrm{d}x\,\mathrm{d}t\\ &\quad+\sum_{i=1}^N\int_{Q_T}a_i(x,t,\nabla u)D^iu_k\,\mathrm{d}x\,\mathrm{d}t+\sum_{i=1}^N\int_{Q_T}a_i(x,t,\nabla u_k)D^iu\,\mathrm{d}x\,\mathrm{d}t\\ &\quad+\gamma\int_{Q_T}|u_k|^{p_0-2}u_ku\,\mathrm{d}x\,\mathrm{d}t+\gamma\int_{Q_T}|u|^{p_0-2}uu_k\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Thus, due to (4.6) and (4.8), we have

$$\liminf_{k\to\infty}\left\{\sum_{i=1}^{N}\int_{Q_T} a_i(x,t,\nabla u_k)D^i u_k\,\mathrm{d}x\,\mathrm{d}t+\gamma\int_{Q_T}|u_k|^{p_0}\,\mathrm{d}x\,\mathrm{d}t\right\} \geqslant \sum_{i=1}^{N}\int_{Q_T}\varphi_i D^i u\,\mathrm{d}x\,\mathrm{d}t+\gamma\int_{Q_T}|u|^{p_0}\,\mathrm{d}x\,\mathrm{d}t.$$

This implies, by (4.12), that

$$\lim_{k\to\infty}\left\{\sum_{i=1}^{N}\int_{Q_T} a_i(x,t,\nabla u_k)D^i u_k\,\mathrm{d}x\,\mathrm{d}t+\gamma\int_{Q_T}|u_k|^{p_0}\,\mathrm{d}x\,\mathrm{d}t\right\} = \sum_{i=1}^{N}\int_{Q_T}\varphi_i D^i u\,\mathrm{d}x\,\mathrm{d}t+\gamma\int_{Q_T}|u|^{p_0}\,\mathrm{d}x\,\mathrm{d}t. \quad (4.14)$$

By combining (4.9), (4.11) and (4.14), we deduce that

$$\int_0^T\langle B_n u_k,u_k\rangle\,\mathrm{d}t\to\int_0^T\langle\chi,u\rangle\,\mathrm{d}t \quad \text{as } k\to+\infty.$$

Now, by (4.14) we can obtain

$$\sum_{i=1}^{N}\int_{Q_T}[a_i(x,t,\nabla u_k)-a_i(x,t,\nabla u)](D^i u_k-D^i u)\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T}(|u_k|^{p_0-2}u_k-|u|^{p_0-2}u)(u_k-u)\,\mathrm{d}x\,\mathrm{d}t\to 0 \quad \text{as } k\to+\infty.$$

By lemma 3.3, we get

$$u_k\to u \quad \text{in } L^{\boldsymbol{p}}(0,T;W_0^{1,\boldsymbol{p}}(Q_T)).$$

Then $D^i u_n\to D^i u$ a.e. in Q_T. Thus, it follows that $a_i(x,t,\nabla u_n)\to a_i(x,t,\nabla u)$ a.e. in Q_T. Hence, in view of (3.1) and lemma 3.2, we have

$$a_i(x,t,\nabla u_n)\rightharpoonup a_i(x,t,\nabla u) \quad \text{in } L^{p_i'}(Q_T) \quad \text{for } i=1,\dots,N.$$

Therefore, using (4.7) and (4.8), we deduce that $\chi=B_n u$, and the proof of lemma 4.2 is completed. □

As a consequence of lemma 4.2, we deduce that there exists at least one weak solution $u_n\in L^{\boldsymbol{p}}(0,T;W_0^{1,\boldsymbol{p}}(\varOmega))$ of problem (4.2) (see [12]).

STEP 2 (*a priori* estimates). Taking u_n as a test function in (4.2), we obtain

$$\int_0^T\left\langle\frac{\partial u_n}{\partial t},u_n\right\rangle\mathrm{d}t+\int_0^T\langle Au_n,u_n\rangle\,\mathrm{d}t + \int_{Q_T}g_n(x,t,u_n)u_n\,\mathrm{d}x\,\mathrm{d}t+\gamma\int_{Q_T}|u_n|^{p_0}\,\mathrm{d}x\,\mathrm{d}t=\int_0^T\langle f,u_n\rangle\,\mathrm{d}t. \quad (4.15)$$

By (3.2), we have

$$\int_0^T \langle Au_n, u_n\rangle \,\mathrm{d}t = \sum_{i=1}^N \int_{Q_T} a_i(x,t,\nabla u_n) D^i u_n \,\mathrm{d}x\,\mathrm{d}t$$
$$\geqslant \alpha \sum_{i=1}^N \int_{Q_T} |D^i u_n|^{p_i} \,\mathrm{d}x\,\mathrm{d}t. \tag{4.16}$$

We also have

$$\int_0^T \left\langle \frac{\partial u_n}{\partial t}, u_n \right\rangle \mathrm{d}t = \frac{1}{2}\int_\Omega \int_0^T \frac{\partial (u_n(t))^2}{\partial t}\,\mathrm{d}t\,\mathrm{d}x$$
$$= \frac{1}{2}\int_\Omega |u_n(T)|^2\,\mathrm{d}x - \frac{1}{2}\int_\Omega |u(0)|^2\,\mathrm{d}x, \tag{4.17}$$

As regards the term on the right-hand side of (4.15), since $f \in L^{\boldsymbol{p}'}(0,T;W_0^{-1,\boldsymbol{p}'}(\Omega))$ there exist $f_0 \in L^{p_0'}(Q_T)$ and $f_i \in L^{p_i'}(Q_T)$ for $i = 1,\dots,N$ such that

$$f = f_0 - \sum_{i=1}^N D^i f_i.$$

Then by using Young's inequality, we get

$$\int_0^T \langle f, u_n\rangle\,\mathrm{d}t = \sum_{i=0}^N \int_{Q_T} f_i D^i u_n \,\mathrm{d}x\,\mathrm{d}t$$
$$\leqslant c_1 \sum_{i=0}^N \int_{Q_T} |f_i|^{p_i'}\,\mathrm{d}x\,\mathrm{d}t + \tfrac{1}{2}\alpha \sum_{i=1}^N \int_{Q_T} |D^i u_n|^{p_i}\,\mathrm{d}x\,\mathrm{d}t$$
$$+ \tfrac{1}{2}\gamma \int_{Q_T} |u_n|^{p_0}\,\mathrm{d}x\,\mathrm{d}t. \tag{4.18}$$

Combining (4.15)–(4.18), we obtain

$$\frac{1}{2}\int_\Omega |u_n(T)|^2\,\mathrm{d}x + \tfrac{1}{2}\alpha \sum_{i=1}^N \int_{Q_T} |D^i u_n|^{p_i}\,\mathrm{d}x\,\mathrm{d}t$$
$$+ \int_{Q_T} g_n(x,t,u_n)u_n\,\mathrm{d}x\,\mathrm{d}t + \tfrac{1}{2}\gamma \int_{Q_T} |u_n|^{p_0}\,\mathrm{d}x\,\mathrm{d}t \leqslant c_2. \tag{4.19}$$

Then there exists a constant c_3 that does not depend on n, such that

$$\sum_{i=0}^N \|D^i u_n\|_{L^{p_i}(Q_T)} \leqslant c_3. \tag{4.20}$$

Therefore, $(u_n)_n$ is bounded in $L^{\boldsymbol{p}}(0,T;W_0^{1,\boldsymbol{p}}(\Omega))$. Hence, there exists a subsequence, still denoted $(u_n)_n$, such that

$$\left.\begin{aligned} &u_n \rightharpoonup u \quad \text{in } L^{\boldsymbol{p}}(0,T;W_0^{1,\boldsymbol{p}}(\Omega)),\\ &u_n \to u \quad \text{in } L^1(Q_T) \text{ and a.e. in } Q_T.\end{aligned}\right\} \tag{4.21}$$

We deduce that

$$|u_n(T)|^2 \to |u(T)|^2 \quad \text{a.e. in } \Omega \quad \text{and} \quad g_n(x,t,u_n)u_n \to g(x,t,u)u \quad \text{a.e. in } Q_T.$$

Then, due to (4.19) and Fatou's lemma, we get

$$\int_\Omega |u(T)|^2 \,\mathrm{d}x \leqslant \liminf_{n\to\infty} \int_\Omega |u_n(T)|^2 \,\mathrm{d}x \tag{4.22}$$

and

$$\int_{Q_T} g(x,t,u)u \,\mathrm{d}x\,\mathrm{d}t \leqslant \liminf_{n\to\infty} \int_{Q_T} g_n(x,t,u_n)u_n \,\mathrm{d}x\,\mathrm{d}t. \tag{4.23}$$

Step 3 (weak convergence of $(u_n)_t$ in $L^{\boldsymbol{p}'}(0,T;W^{-1,\boldsymbol{p}'}(\Omega))$). Taking $v \in L^{\boldsymbol{p}}(0,T;W_0^{1,\boldsymbol{p}}(\Omega))$ as a test function in (4.2), we get

$$\int_0^T \left\langle \frac{\partial u_n}{\partial t}, v\right\rangle \mathrm{d}t + \int_0^T \langle Au_n, v\rangle \,\mathrm{d}t + \int_{Q_T} g_n(x,t,u_n)v \,\mathrm{d}x\,\mathrm{d}t + \gamma \int_{Q_T} |u_n|^{p_0-2}u_n v \,\mathrm{d}x\,\mathrm{d}t = \int_0^T \langle f, v\rangle \,\mathrm{d}t.$$

Then,

$$\left| \int_0^T \left\langle \frac{\partial u_n}{\partial t}, v\right\rangle \mathrm{d}t\right| \leqslant \left| \int_0^T \langle Au_n, v\rangle \,\mathrm{d}t\right| + \int_{Q_T} |g_n(x,t,u_n)||v| \,\mathrm{d}x\,\mathrm{d}t + \gamma \int_{Q_T} |u_n|^{p_0-1}|v| \,\mathrm{d}x\,\mathrm{d}t + \left| \int_0^T \langle f, v\rangle \,\mathrm{d}t\right|. \tag{4.24}$$

For the first term on the right-hand side of (4.24), using Hölder's inequality and (4.20), we get

$$\begin{aligned}
&\left| \int_0^T \langle Au_n, v\rangle \,\mathrm{d}t\right| \\
&\quad \leqslant \sum_{i=1}^N \int_{Q_T} |a_i(x,t,\nabla u_n)||D^i v| \,\mathrm{d}x\,\mathrm{d}t \\
&\quad \leqslant \sum_{i=1}^N \int_{Q_T} \beta(K_i(x,t) + |D^i u_n|^{p_i-1})|D^i v| \,\mathrm{d}x\,\mathrm{d}t \\
&\quad \leqslant \beta \sum_{i=1}^N (\|K_i(x,t)\|_{L^{p_i'}(Q_T)} + \|D^i u_n\|_{L^{p_i}(Q_T)}^{p_i-1})\|D^i v\|_{L^{p_i}(Q_T)} \\
&\quad \leqslant \beta\bigg(\sum_{i=1}^N \|K_i(x,t)\|_{L^{p_i'}(Q_T)} + \sum_{i=1}^N \|D^i u_n\|_{L^{p_i}(Q_T)}^{p_i-1}\bigg) \sum_{i=1}^N \|D^i v\|_{L^{p_i}(Q_T)} \\
&\quad \leqslant c_4 \|v\|_{L^{\boldsymbol{p}}(0,T;W_0^{1,\boldsymbol{p}}(\Omega))}.
\end{aligned} \tag{4.25}$$

Regarding the second and third terms on the left-hand side of (4.24), due to (3.4) we have

$$\begin{aligned}\int_{Q_T} |g(x,t,u_k)||v|\,\mathrm{d}x\,\mathrm{d}t &+ \gamma \int_{Q_T} |u_n|^{p_0-1}|v|\,\mathrm{d}x\,\mathrm{d}t \\ &\leqslant (1+\gamma)\int_{Q_T} |u_k|^{p_0-1}|v|\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} b(x,t)|v|\,\mathrm{d}x\,\mathrm{d}t \\ &\leqslant ((1+\gamma)\||u_k|^{p_0-1}\|_{L^{p_0'}(Q_T)} + \|b(x,t)\|_{L^{p_0'}(Q_T)})\|v\|_{L^{p_0}(Q_T)} \\ &\leqslant c_5\|v\|_{L^{\boldsymbol{p}}(0,T;W_0^{1,\boldsymbol{p}}(\Omega))}. \end{aligned} \tag{4.26}$$

Concerning the last term on the right-hand side of (4.24), we have

$$\begin{aligned}\left|\int_0^T \langle f, v\rangle\,\mathrm{d}t\right| \leqslant \sum_{i=0}^N \int_{Q_T} |f_i||D^i v|\,\mathrm{d}x\,\mathrm{d}t &\leqslant \sum_{i=0}^N \|f_i\|_{L^{p_i'}(Q_T)}\|D^i v\|_{L^{p_i}(Q_T)} \\ &\leqslant \|f\|_{L^{\boldsymbol{p}'}(0,T;W^{1,\boldsymbol{p}'}(\Omega))}\|v\|_{L^{\boldsymbol{p}}(0,T;W_0^{1,\boldsymbol{p}}(\Omega))}. \end{aligned} \tag{4.27}$$

Consequently, by combining (4.24)–(4.27), we deduce that

$$\left|\int_0^T \left\langle \frac{\partial u_n}{\partial t}, v\right\rangle \mathrm{d}t\right| \leqslant c_6\|v\|_{L^{\boldsymbol{p}}(0,T;W_0^{1,\boldsymbol{p}}(\Omega))}, \tag{4.28}$$

where c_6 is a constant that does not depend on n. Then $\partial u_n/\partial t$ is bounded in $L^{\boldsymbol{p}'}(0,T;W^{-1,\boldsymbol{p}'}(\Omega))$ and

$$\frac{\partial u_k}{\partial t} \rightharpoonup \frac{\partial u}{\partial t} \quad \text{in } L^{\boldsymbol{p}'}(0,T;W^{-1,\boldsymbol{p}'}(\Omega)).$$

Step 4 (convergence of the gradient). By taking $u_n - u$ as a test function in (4.2), we obtain

$$\begin{aligned}\int_0^T \left\langle \frac{\partial u_n}{\partial t}, u_n - u\right\rangle \mathrm{d}t &+ \int_0^T \langle Au_n, u_n - u\rangle\,\mathrm{d}t \\ &+ \int_{Q_T} g_n(x,t,u_n)(u_n-u)\,\mathrm{d}x\,\mathrm{d}t + \gamma\int_{Q_T} |u_n|^{p_0-2}u_n(u_n-u)\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_0^T \langle f, u_n - u\rangle\,\mathrm{d}t. \end{aligned}$$

Hence,

$$\begin{aligned}\int_0^T \langle Au_n - Au, u_n - u\rangle\,\mathrm{d}t &+ \gamma\int_{Q_T} (|u_n|^{p_0-2}u_n - |u|^{p_0-2}u)(u_n-u)\,\mathrm{d}x\,\mathrm{d}t \\ = -\int_0^T \left\langle \frac{\partial u_n}{\partial t}, u_n - u\right\rangle \mathrm{d}t &- \int_{Q_T} g_n(x,t,u_n)(u_n-u)\,\mathrm{d}x\,\mathrm{d}t \\ + \int_0^T \langle f, u_n - u\rangle\,\mathrm{d}t &- \int_0^T \langle Au, u_n-u\rangle\,\mathrm{d}t - \gamma\int_{Q_T} |u|^{p_0-2}u(u_n-u)\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{4.29}$$

Now, we shall study the first and second terms on the right-hand side of (4.29). Due to the fact that

$$\frac{\partial u_n}{\partial t} \rightharpoonup \frac{\partial u}{\partial t} \quad \text{in } L^{\boldsymbol{p}'}(0,T;W^{-1,\boldsymbol{p}'}(\Omega))$$

and due to (4.22), we obtain

$$\begin{aligned}
&\liminf_{n\to\infty} \int_0^T \left\langle \frac{\partial u_n}{\partial t}, u_n - u \right\rangle \mathrm{d}t \\
&\qquad = \liminf_{n\to\infty} \int_\Omega \int_0^T \frac{\partial u_n}{\partial t}(u_n - u)\,\mathrm{d}t\,\mathrm{d}x \\
&\qquad = \liminf_{n\to\infty} \int_\Omega \int_0^T \frac{\partial u_n}{\partial t} u_n\,\mathrm{d}t\,\mathrm{d}x - \int_\Omega \int_0^T \frac{\partial u}{\partial t} u\,\mathrm{d}t\,\mathrm{d}x \\
&\qquad = \liminf_{n\to\infty} \int_\Omega \left[\frac{u_n^2(T) - u_n^2(0)}{2}\right] \mathrm{d}x - \int_\Omega \left[\frac{u^2(T) - u^2(0)}{2}\right] \mathrm{d}x \\
&\qquad = \liminf_{n\to\infty} \int_\Omega \frac{u_n^2(T)}{2}\,\mathrm{d}x - \int_\Omega \frac{u^2(T)}{2}\,\mathrm{d}x \\
&\qquad \geqslant 0,
\end{aligned}$$

since $u_n(0) = u(0)$. Thus,

$$\limsup_{n\to\infty} - \int_0^T \left\langle \frac{\partial u_n}{\partial t}, u_n - u \right\rangle \mathrm{d}t = - \liminf_{n\to\infty} \int_0^T \left\langle \frac{\partial u_n}{\partial t}, u_n - u \right\rangle \mathrm{d}t \leqslant 0. \tag{4.30}$$

On the other hand, we get that $g_n(x,t,u_n)$ is bounded in $L^{p_0'}(Q_T)$ and

$$g_n(x,t,u_n) \to g(x,t,u) \quad \text{a.e. in } Q_T.$$

Hence, in view of lemma 3.2, we get

$$g_n(x,t,u_n) \rightharpoonup g(x,t,u) \quad \text{in } L^{p_0'}(Q_T).$$

Then, by (4.23), we also get

$$\begin{aligned}
&\liminf_{n\to\infty} \int_{Q_T} g_n(x,t,u_n)(u_n - u)\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad = \liminf_{n\to\infty} \int_{Q_T} g_n(x,t,u_n)u_n\,\mathrm{d}x\,\mathrm{d}t - \int_{Q_T} g(x,t,u)u\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad \geqslant 0.
\end{aligned}$$

Consequently,

$$\limsup_{n\to\infty} - \int_{Q_T} g_n(x,t,u_n)(u_n - u)\,\mathrm{d}x\,\mathrm{d}t \leqslant 0. \tag{4.31}$$

By combining (4.29)–(4.31), we deduce that

$$\begin{aligned}0 \leqslant \limsup_{n\to\infty}\biggl(\int_0^T \langle Au_n - Au, u_n - u\rangle \,\mathrm{d}t \\ + \gamma \int_{Q_T} (|u_n|^{p_0-2}u_n - |u|^{p_0-2}u)(u_n - u)\,\mathrm{d}x\,\mathrm{d}t\biggr) \\ \leqslant \lim_{n\to\infty}\int_0^T \langle f, u_n - u\rangle \,\mathrm{d}t - \lim_{n\to\infty}\int_0^T \langle Au, u_n - u\rangle \,\mathrm{d}t \\ - \gamma \lim_{n\to\infty}\int_{Q_T} |u|^{p_0-2}u(u_n - u)\,\mathrm{d}x\,\mathrm{d}t. \qquad (4.32)\end{aligned}$$

Now, in view of (4.21), it is clear that, as $n \to \infty$,

$$\int_0^T \langle f, u_n - u\rangle \,\mathrm{d}t \to 0, \qquad (4.33)$$

$$\int_0^T \langle Au, u_n - u\rangle \,\mathrm{d}t = \sum_{i=1}^N \int_{Q_T} a_i(x,t,\nabla u)(D^i u_n - D^i u)\,\mathrm{d}x\,\mathrm{d}t \to 0, \qquad (4.34)$$

$$\int_{Q_T} |u|^{p_0-2}u(u_n - u)\,\mathrm{d}x\,\mathrm{d}t \to 0. \qquad (4.35)$$

Hence, it follows that

$$\begin{aligned}\int_0^T \langle Au_n - Au, u_n - u\rangle \,\mathrm{d}t \\ + \gamma \int_{Q_T} (|u_n|^{p_0-2}u_n - |u|^{p_0-2}u)(u_n - u)\,\mathrm{d}x\,\mathrm{d}t \to 0 \quad \text{as } n \to \infty. \qquad (4.36)\end{aligned}$$

Thus, due to lemma 3.3, we deduce that

$$u_n \to u \quad \text{in } L^{\boldsymbol{p}}(0,T; W_0^{1,\boldsymbol{p}}(\Omega)). \qquad (4.37)$$

STEP 5 (equi-integrability of $g_n(x,t,u_n)$). In order to prove that

$$g_n(x,t,u_n) \to g(x,t,u) \quad \text{strongly in } L^1(Q_T)$$

using Vitali's theorem, it is sufficient to prove that $g_n(x,t,u_n)$ is uniformly equi-integrable. Indeed, let $m > 0$ and let E be a measurable subset of Q_T. Then, we have

$$\begin{aligned}\int_E |g_n(x,t,u_n)|\,\mathrm{d}x\,\mathrm{d}t \\ \leqslant \int_{E\cap\{|u_n|\leqslant m\}} |g_n(x,t,u_n)|\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{m}\int_{E\cap\{|u_n|>m\}} g_n(x,t,u_n)u_n\,\mathrm{d}x\,\mathrm{d}t \\ \leqslant \int_{E\cap\{|u_n|\leqslant m\}} (|u_n|^{p_0-1} + b(x,t))\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{m}\int_{Q_T} g_n(x,t,u_n)u_n\,\mathrm{d}x\,\mathrm{d}t \\ \leqslant |m|^{p_0-1}|E| + \int_E b(x,t)\,\mathrm{d}x\,\mathrm{d}t + \frac{c_2}{m}, \qquad (4.38)\end{aligned}$$

where c_2 is the constant of (4.19), which is independent of n. Then, for all $\varepsilon > 0$, there exist m large enough such that $c_2/m < \varepsilon/2$ and $|E|$ sufficiently small to obtain

$$|m|^{p_0-1}|E| + \int_E b(x,t)\,\mathrm{d}x\,\mathrm{d}t < \tfrac{1}{2}\varepsilon.$$

Thus,

$$\int_E |g_n(x,t,u_n)|\,\mathrm{d}x\,\mathrm{d}t \leqslant \varepsilon.$$

Using Vitali's theorem, we obtain

$$g_n(x,t,u_n) \to g(x,t,u) \quad \text{in } L^1(Q_T). \tag{4.39}$$

Moreover, in view of (4.19), we have

$$\int_{Q_T} g(x,t,u)u\,\mathrm{d}x\,\mathrm{d}t \leqslant \liminf_{n\to\infty} \int_{Q_T} g_n(x,t,u_n)u_n\,\mathrm{d}x\,\mathrm{d}t \leqslant c_2.$$

Therefore, $g(x,t,u)u \in L^1(Q_T)$.

Step 6 (passing to the limit). Let us take $v \in L^{\boldsymbol{p}}(0,T;W_0^{1,\boldsymbol{p}}(\Omega))$ as a test function in (4.2). We obtain

$$\int_0^T \left\langle \frac{\partial u_n}{\partial t}, v\right\rangle \mathrm{d}t + \int_0^T \langle Au_n, v\rangle\,\mathrm{d}t + \int_{Q_T} g_n(x,t,u_n)v\,\mathrm{d}x\,\mathrm{d}t + \gamma \int_{Q_T} |u_n|^{p_0-2}u_n v\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \langle f, v\rangle\,\mathrm{d}t. \tag{4.40}$$

It is clear to see that

$$\int_0^T \left\langle \frac{\partial u_n}{\partial t}, v\right\rangle \mathrm{d}t \to \int_0^T \left\langle \frac{\partial u}{\partial t}, v\right\rangle \mathrm{d}t \quad \text{as } n\to\infty. \tag{4.41}$$

By (4.37), we have $a_i(x,t,\nabla u_n) \to a_i(x,t,\nabla u)$ a.e. in Q_T, and $a_i(x,t,\nabla u_n)$ is bounded in $L^{p_i'}(Q_T)$. Then, in view of lemma 3.2, we get

$$a_i(x,t,\nabla u_n) \rightharpoonup a_i(x,t,\nabla u) \text{ in } L^{p_i'}(Q_T) \quad \text{for } i = 1,\dots,N.$$

Consequently, as $n\to\infty$,

$$\int_0^T \langle Au_n, v\rangle\,\mathrm{d}t = \sum_{i=1}^N \int_{Q_T} a_i(x,t,\nabla u_n)D^i v\,\mathrm{d}x\,\mathrm{d}t \to \sum_{i=1}^N \int_{Q_T} a_i(x,t,\nabla u)D^i v\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \langle Au, v\rangle\,\mathrm{d}t, \tag{4.42}$$

$$\int_{Q_T} g_n(x,t,u_n)v\,\mathrm{d}x\,\mathrm{d}t \to \int_{Q_T} g(x,t,u)v\,\mathrm{d}x\,\mathrm{d}t, \tag{4.43}$$

and

$$\int_{Q_T} |u_n|^{p_0-2}u_n v\,\mathrm{d}x\,\mathrm{d}t \to \int_{Q_T} |u|^{p_0-2}uv\,\mathrm{d}x\,\mathrm{d}t. \tag{4.44}$$

Then, passing to the limit in (4.40), we obtain

$$\int_0^T \left\langle \frac{\partial u}{\partial t}, v \right\rangle \mathrm{d}t + \int_0^T \langle Au, v\rangle \,\mathrm{d}t \\ + \int_{Q_T} g(x,t,u)v \,\mathrm{d}x\,\mathrm{d}t + \gamma \int_{Q_T} |u|^{p_0-2}uv \,\mathrm{d}x\,\mathrm{d}t = \int_0^T \langle f, v\rangle \,\mathrm{d}t \tag{4.45}$$

for all $u, v \in L^{\boldsymbol{p}}(0,T; W_0^{1,\boldsymbol{p}}(\Omega))$.

This completes the proof. □

PROPOSITION 4.3. *Assume, furthermore, that*

$$(g(x,t,s) - g(x,t,r))(s-r) \geqslant 0 \text{ a.e. in } \Omega \quad \text{for any } s, r \in \mathbb{R}. \tag{4.46}$$

Then, the weak solution of problem (4.1) is unique.

Proof. Let $u_1, u_2 \in L^{\boldsymbol{p}}(0,T; W_0^{1,\boldsymbol{p}}(\Omega))$ be two weak solutions of (4.1). Then, we have

$$\int_0^T \left\langle \frac{\partial u_1}{\partial t} - \frac{\partial u_2}{\partial t}, v \right\rangle \mathrm{d}t + \sum_{i=1}^N \int_{Q_T} (a_i(x,t,\nabla u_1) - a_i(x,t,\nabla u_2))D^i v \,\mathrm{d}x\,\mathrm{d}t \\ + \int_{Q_T} (g(x,t,u_1) - g(x,t,u_2))v \,\mathrm{d}x\,\mathrm{d}t \\ + \gamma \int_{Q_T} (|u_1|^{p_0-2}u_1 - |u_2|^{p_0-2}u_2)v \,\mathrm{d}x\,\mathrm{d}t = 0$$

for any $v \in L^{\boldsymbol{p}}(0,T; W_0^{1,\boldsymbol{p}}(\Omega))$. In particular, by taking $v = u_1 - u_2$ we obtain

$$0 \leqslant \sum_{i=1}^N \int_{Q_T} (a_i(x,t,\nabla u_1) - a_i(x,t,\nabla u_2))(D^i u_1 - D^i u_2) \,\mathrm{d}x\,\mathrm{d}t \\ + \gamma \int_{Q_T} (|u_1|^{p_0-2}u_1 - |u_2|^{p_0-2}u_2)(u_1 - u_2) \,\mathrm{d}x\,\mathrm{d}t \\ = -\int_{Q_T} (g(x,t,u_1) - g(x,t,u_2))(u_1 - u_2) \,\mathrm{d}x\,\mathrm{d}t - \int_0^T \left\langle \frac{\partial u_1}{\partial t} - \frac{\partial u_2}{\partial t}, u_1 - u_2 \right\rangle \mathrm{d}t. \tag{4.47}$$

By (4.46), we get

$$\int_{Q_T} (g(x,t,u_1) - g(x,t,u_2))(u_1 - u_2) \,\mathrm{d}x\,\mathrm{d}t \geqslant 0.$$

Now, since $u_1(x,0) = u_2(x,0) = u_0(x)$,

$$\int_0^T \left\langle \frac{\partial u_1}{\partial t} - \frac{\partial u_2}{\partial t}, u_1 - u_2 \right\rangle \mathrm{d}t = \int_\Omega \int_0^T \frac{\partial (u_1 - u_2)^2}{\partial t} \,\mathrm{d}t\,\mathrm{d}x \\ = \int_\Omega (u_1(T) - u_2(T))^2 \,\mathrm{d}x \\ \geqslant 0.$$

Therefore, due to (4.47), we conclude that

$$\sum_{i=1}^{N}\int_{Q_T}(a_i(x,t,\nabla u_1)-a_i(x,t,\nabla u_2))(D^iu_1-D^iu_2)\,\mathrm{d}x\,\mathrm{d}t + \gamma\int_{Q_T}(|u_1|^{p_0-2}u_1-|u_2|^{p_0-2}u_2)(u_1-u_2)\,\mathrm{d}x\,\mathrm{d}t=0. \quad (4.48)$$

Then $u_1=u_2$ a.e. in Q_T, which concludes our proof. □

Remark 4.4. If the hypothesis (4.46) is not supported, the uniqueness of the weak solution of the problem (4.1) is not guaranteed.

Proposition 4.5. *Let* $f\in L^{\boldsymbol{p}'}(0,T;W^{-1,\boldsymbol{p}'}(\Omega))$ *and assume that (3.1)–(3.5) hold. Moreover, assume that*

$$g(x,t,s)s\geqslant\delta|s|^{p_0} \quad (4.49)$$

with $\delta>0$. *Then, the problem*

$$\left.\begin{aligned} u_t+Au+g(x,t,u)&=f && \text{in } Q_T=\Omega\times(0,T),\\ u(x,t)&=0 && \text{on } S_T=\partial\Omega\times(0,T),\\ u(x,0)&=u_0(x) && \text{in } \Omega, \end{aligned}\right\} \quad (4.50)$$

has at least one weak solution $u\in L^{\boldsymbol{p}}(0,T;W_0^{1,\boldsymbol{p}}(\Omega))$.

Indeed, we set $g(x,t,s)=h(x,t,s)+\delta|s|^{p_0-2}s$. Then

$$h(x,t,s)s=g(x,t,s)s-\delta|s|^{p_0}\geqslant 0 \quad \text{and} \quad |h(x,t,s)|\leqslant|g(x,t,s)|.$$

Therefore, the existence of weak solutions follows immediately from theorem 4.1.

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