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# NONLINEAR PARABOLIC BOUNDARY VALUE PROBLEMS OF INFINITE ORDER

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Abstract. In this paper an existence result is presented for solution of a parabolic boundary value problem under Dirichlet null boundary conditions for a class of general equations of infinite order with strongly nonlinear perturbation terms.

## 1. Introduction

Let $\Omega$ be a bounded open set of $\mathbb{R}^N$, $Q = [0,T] \times \Omega$ be a cylinder with lateral surface $S = [0,T] \times \Gamma$, where $\Gamma$ is the boundary of $\Omega$.

Our purpose is to study, in the cylinder $Q$, the following strongly nonlinear parabolic problem of Dirichlet type:

$$\text{(P)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} + Au + g(t,x,u) = f(t,x), \\ u(0,x) = 0, \\ D^\omega u = 0, \quad \text{on } S \text{ for all } |\omega| = 0,1,\dots, \end{cases}$$

where $A$ is a nonlinear elliptic operator of infinite order defined by

$$Au = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} D^\alpha (A_\alpha(t,x,D^\gamma u)). \tag{1.1}$$

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The real functions $A_\alpha(t,x,\xi)$ are required to have polynomial growth in $\xi$, for all multi-indices $\alpha$ and $g$ is a nonlinear term which has to fulfil a sign condition.

In the case of infinite order, Dubinskiĭ [7] has proved, under some growth and certain monotonicity conditions, the existence of solutions for the Dirichlet problem associated with the equation $Au = f$ in some general functional Sobolev spaces of infinite order $W_0^\infty(a_\alpha, p_\alpha)$, with variable exponents $p_\alpha$, where $\alpha$ is a multi-indice. The same author has investigated the existence result for parabolic elliptic problems governed by operators of infinite orders. In fact, also in [7], Dubinskiĭ has proved by considering, further, the monotonicity of the operator $A$ that the problem $\frac{\partial u}{\partial t} + Au = f$ has a solution in $L^p(0,T,W_0^\infty(a_\alpha,p))$, $p > 1$, in the dual case. Our purpose in this paper is to prove the existence of solutions for strongly parabolic nonlinear equations of infinite order related to the problem $\frac{\partial u}{\partial t} + Au + g(t,x,u) = f$. More precisely, we will assume more less restrictions on the operator $A$ (no monotonicity condition) and deal with a different approach by involving a truncation of the perturbations $g$. Next, we use the monotonicity of a part of the approximate operator which contains a linear term of higher order of derivation that satisfies the monotonicity condition and prove the existence of solutions in the framework of function spaces $L^p(0,T,W_0^\infty(a_\alpha,p))$, $p > 1$.

Let us mention that an interesting result concerning the stationary counterpart of the problem (P) has been proved in [2].

## 2. Preliminaries

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$, $a_\alpha \geq 0$, $p > 1$ are real numbers for all multi-indices $\alpha$, and $\|\cdot\|_p$ is the usual Lebesgue norm in the space $L^p(\Omega)$. The Sobolev space of infinite order is the functional space defined by

$$W^\infty(a_\alpha,p)(\Omega) = \bigg\{u \in C^\infty(\Omega) : \|u\|_\infty^p = \sum_{|\alpha|=0}^{\infty} a_\alpha \|D^\alpha u\|_p^p \ < \infty\bigg\}.$$

Here

$$D^\alpha = \frac{\partial^{|\alpha|}}{(\partial x_1)^{\alpha_1}\ldots(\partial x_N)^{\alpha_N}}.$$

We denote by $C_0^\infty(\Omega)$ the space of all functions with compact support in $\Omega$ with continuous derivatives of arbitrary order.

Since we shall deal with the Dirichlet problem, we will use the functional space $W_0^\infty(a_\alpha,p)(\Omega)$ defined by

$$W_0^\infty(a_\alpha,p)(\Omega) = \bigg\{u \in C_0^\infty(\Omega) : \|u\|_\infty^p = \sum_{|\alpha|=0}^{\infty} a_\alpha \|D^\alpha u\|_p^p < \infty\bigg\}.$$

In contrast with the finite order Sobolev space, the very first question, which arises in the study of $W^{\infty}(a_\alpha, p)(\Omega)$, is the question of their nontriviality (or nonemptiness), i.e. the question of the existence of a function $u$ such that $\|u\|_\infty < \infty$.

Definition 2.1 ([7]). The space $W_0^{\infty}(a_\alpha, p)(\Omega)$ is called nontrivial space if it contains at least one function which not identically equal to zero, i.e. there is a function $u \in C_0^{\infty}(\Omega)$ such that $\|u\|_\infty < \infty$.

It turns out that the answer of this question depends not only on the given parameters $a_\alpha$ and $p$ of the spaces $W^{\infty}(a_\alpha, p)(\Omega)$, but also on the domain $\Omega$.

The dual space of $W_0^{\infty}(a_\alpha, p)(\Omega)$ is defined as follows

$$W^{-\infty}(a_\alpha, p')(\Omega) = \bigg\{ f : f = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} D^\alpha f_\alpha,\ \|f\|_{-\infty}^{p'} = \sum_{|\alpha|=0}^{\infty} a_\alpha \|f_\alpha\|_{p'}^{p'} < \infty \bigg\},$$

where $f_\alpha \in L^{p'}(\Omega)$ for all multi-indices $\alpha$ and $p'$ is the conjugate of $p$, i.e. $p' = p/(p-1)$ (for more details about these spaces, see [7] and [8]).

By the definition, the duality of $W^{-\infty}(a_\alpha, p')(\Omega)$ and $W_0^{\infty}(a_\alpha, p)(\Omega)$ is given by the relation

$$\langle f, v \rangle = \sum_{|\alpha|=0}^{\infty} a_\alpha \int_\Omega f_\alpha(x) D^\alpha v(x)\, dx,$$

which, as it is not difficult to verify, is correct.

Let us denote by $L^p(0, T, W_0^{\infty}(a_\alpha, p))$ the space of functions $u(t,x)$ which has finite norm

$$\|u\|_{p,\infty} = \bigg( \int_0^T \|u\|_\infty^p\, dt \bigg)^{1/p}$$

and are equal to zero together with all derivatives $D^\omega u$ on the lateral surface $S$. In other word one has

$$L^p(0, T, W_0^{\infty}(a_\alpha, p)) = \bigg\{ u(t,x) : \|u\|_{p,\infty}^p = \sum_{|\alpha|=0}^{\infty} a_\alpha \int_0^T \|D^\alpha u\|_p^p\, dt < \infty, \\ D^\omega u|_S = 0, |\omega| = 0, 1, \ldots \bigg\}.$$

Further, let $L^{p'}(0, T, W^{-\infty}(a_\alpha, p'))$ be the dual space of $L^p(0, T, W_0^{\infty}(a_\alpha, p))$, that is, the space of generalized functions $f(t,x)$ having a form

$$f(t,x) = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha D^\alpha f_\alpha(t,x),$$

where $f_\alpha(t,x) \in L^{p'}(Q)$ and

$$\rho'(f) = \sum_{|\alpha|=0}^{\infty} a_\alpha \int_0^T \|f_\alpha(t,x)\|_{p'}^{p'}\, dt < \infty.$$

The value of $f(t,x) \in L^{p'}(0,T,W^{-\infty}(a_\alpha,p'))$ on an element $v(t,x) \in L^p(0,T, W_0^\infty(a_\alpha,p))$ is defined by the formula

$$\int_0^T \langle f,v\rangle\,dt = \sum_{|\alpha|=0}^{\infty} a_\alpha \int_0^T \int_\Omega f_\alpha(t,x) D^\alpha v(t,x)\,dx\,dt,$$

which, as easy to see, is correct.

Sobolev spaces of infinite order have extensive applications to the theory of partial differential equations and, among their number, in mathematical physics. The basis of these applications is the non-formal algebra of differential operators of infinite orders as the operators, acting in the corresponding Sobolev spaces of infinite order. This makes it possible, by considering $\frac{\partial}{\partial x}$ as a parameter, to solve a partial equation as ordinary differential equation, to which are adjoined the initial or boundary conditions.

More explicitly, we cite the following examples of operators of infinite order which are closely inspired from the ones used in Dubinskiĭ [7].

Example 2.2. Let us consider the operator

$$Au = [\cos D]u(x), \quad x \in \mathbb{R}^N,\ N \geq 2.$$

Formally we have

$$Au(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n!} D^{2n} u(x).$$

The functional space corresponding to the Dirichlet type problem, is the Sobolev space of infinite order $W_0^\infty(1/(2n!),2)(\mathbb{R}^N)$, which is nontrivial.

Consequently, if $f \in W^{-\infty}(1/(2n!),2)(\mathbb{R}^N)$, then there exists a weak solution $u \in W_0^\infty(1/(2n!),2)(\mathbb{R}^N)$ for the problem

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n!} D^{2n} u = f, \quad x \in \mathbb{R}^N.$$

Moreover, for any $f \in L^2(0,T,W^{-\infty}(1/(2n!),2)(\mathbb{R}^N))$, the parabolic problem

$$\frac{\partial u}{\partial t} + \sum_{n=0}^{\infty} \frac{(-1)^n}{2n!} D^{2n} u = f, \quad x \in \mathbb{R}^N,$$

has a solution $u \in L^2(0,T,W_0^\infty(1/(2n!),2)(\mathbb{R}^N))$, in the variational sense.

Example 2.3. In the half plane $\mathbb{R}^+ = \{t > 0,\ x \in \mathbb{R}\}$ we consider the heat equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 \tag{2.1}$$

under the initial condition

$$u(0,x) = \varphi(x). \tag{2.2}$$

Putting $p = \frac{\partial^2}{\partial x^2}$, we have

$$u(t,x) = e^{tp} c(x),$$

where $c(x)$ is an arbitrary function. From the condition (2.2) it follows that $c(x) = \varphi(x)$ and consequently, the desired solution has the form

$$u(t,x) = e^{t\frac{\partial^2}{\partial x^2}} \varphi(x). \tag{2.3}$$

For any $t > 0$ the operator

$$e^{t\frac{\partial^2}{\partial x^2}} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{\partial^{2n}}{\partial x^{2n}}, \tag{2.4}$$

is an elliptic differential operator of infinite order. It is obvious that the formula (2.4) is correct in $L^2(\mathbb{R})$ for any $\varphi(x) \in W^\infty$, where

$$W^\infty = \left\{ \varphi \in C_0^\infty(\mathbb{R}) : \sum_{n=0}^{\infty} \frac{t^n}{n!} \|D^{2n}\varphi\|_2 < \infty \right\}.$$

The nontriviality of the functional space $W^\infty$ follows from Theorem 2.1, Chapter I in [7].

## 3. Main result

In this section we formulate and prove the main result. Let $A$ be the nonlinear operator of infinite order defined in (1.1) satisfying:

(A$_1$) $A_\alpha(t,x,\xi_\gamma)$ is a Carathéodory function for all $\alpha$, $|\gamma| \le |\alpha|$.

(A$_2$) For almost every $(t,x) \in Q$, all $m \in \mathbb{N}^*$, all $\xi_\gamma$, $\eta_\alpha$, $|\gamma| \le |\alpha|$ and some constant $c_0 > 0$, we assume that

$$\left| \sum_{|\alpha|=0}^{m} A_\alpha(t,x,\xi_\gamma)\eta_\alpha \right| \le c_0 \sum_{|\alpha|=0}^{m} a_\alpha |\xi_\alpha|^{p-1} |\eta_\alpha|,$$

where $p > 1$, $a_\alpha \ge 0$ are reals numbers for all multi-indices $\alpha$.

(A$_3$) There exist constants $c_1 > 0, c_2 \ge 0$ such that

$$\sum_{|\alpha|=0}^{m} A_\alpha(t,x,\xi_\gamma)\xi_\alpha \ge c_1 \sum_{|\alpha|=0}^{m} a_\alpha |\xi_\alpha|^p - c_2,$$

for all $m \in \mathbb{N}^*$, for all $\xi_\gamma$, $\xi_\alpha$; $|\gamma| \le |\alpha|$.

(A$_4$) The space $W_0^\infty(a_\alpha, p)(\Omega)$ is nontrivial.

As regard to the nonlinear term $g$, we assume that $g$ satisfies the following natural growth on $|u|$ and the classical sign condition:

(G) $g\colon Q \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying

$$|g(t,x,s)| \le |s|^{p-1} + 1, \qquad g(t,x,s)s \ge 0$$

for almost every $(t,x) \in Q$ and all $s \in \mathbb{R}$.

Theorem 3.1. *Under assumptions* $(\mathrm{A}_1)$–$(\mathrm{A}_4)$ *and* (G), *for any* $f \in L^{p'}(0,T, W^{-\infty}(a_\alpha,p'))$, *there exists* $u \in L^p(0,T,W_0^\infty(a_\alpha,p))$ *such that*

$$\int_0^T \left\langle \frac{\partial u}{\partial t}, v \right\rangle dt + \sum_{|\alpha|=0}^{\infty} a_\alpha \int_0^T \int_\Omega A_\alpha(t,x,D^\gamma u) D^\alpha v \, dx \, dt + \int_0^T \int_\Omega g(t,x,u) v \, dx \, dt = \int_0^T \langle f, v \rangle \, dt,$$

*for all* $v \in L^p(0,T,W_0^\infty(a_\alpha,p))$.

Proof. Set, for almost every $(t,x) \in Q$, $g_k(t,x,u) = T_k g(t,x,u)$, where $T_k$ is the usual truncation given by

$$T_k \eta = \begin{cases} \eta & \text{if } |\eta| \le k, \\ \dfrac{k\eta}{|\eta|} & \text{if } |\eta| > k, \end{cases}$$

and let the operator of order $2k+2$ defined by

$$A_{2k+2} u = \sum_{|\alpha|=k+1} (-1)^{k+1} c_\alpha D^{2\alpha} u + \sum_{|\alpha|=0}^{k} (-1)^{|\alpha|} D^\alpha (A_\alpha(t,x,D^\gamma u)).$$

Note that $c_\alpha$ are constants small enough such that they fulfil the conditions of the following lemma introduced in [7]. In fact, such a condition imposed on each $c_\alpha$ is required to ensure the non-triviality of the space $W_0^\infty(c_\alpha,2)$.

Lemma 3.2 (cf. [7]). *For any nontrivial space* $W_0^\infty(a_\alpha,p_\alpha)$, *there exists a nontrivial space* $W_0^\infty(c_\alpha,2)$ *such that* $W_0^\infty(a_\alpha,p_\alpha) \subset W_0^\infty(c_\alpha,2)$.

The operator $A_{2k+2}$ is clearly monotone since the term of higher order of derivation is linear and satisfies the monotonicity condition (see [2] and [7]). Moreover, as in [15], thanks to the truncation $T_k$ and from assumptions $(\mathrm{A}_1)$–$(\mathrm{A}_3)$, we deduce that the operator $A_{2k+2} + g_k$ is bounded, coercive and pseudo-monotone. Then, it is well known (see J.L. Lions [14]), that there exists $u_k \in L^p(0,T,W_0^{k+1,p}(\Omega))$ such that

$$(\mathrm{P}_k) \qquad \begin{cases} \dfrac{\partial u_k}{\partial t} + A_{2k+2} u_k + g_k(t,x,u_k) = f_k(t,x), \\ u_k(0,x) = 0 \end{cases}$$

where

$$f_k(t,x) = \sum_{|\alpha|=0}^{k} (-1)^{|\alpha|} a_\alpha D^\alpha f_\alpha(t,x),$$

with $f_\alpha \in L^{p'}(Q)$ for all $|\alpha| \le k$. In the variational formulation, we get

$$\int_0^T \left\langle \frac{\partial u_k}{\partial t}, v \right\rangle dt + \int_0^T \langle A_{2k+2} u_k, v \rangle \, dt + \int_0^T \int_\Omega g_k(t,x,u_k) v \, dx \, dt = \int_0^T \langle f_k, v \rangle \, dt,$$

for any $v \in L^p(0,T,W_0^{k+1}(\Omega))$.

Let us choose $v = u_k$ as a test function. Using the sign condition, one has the estimates

$$\sum_{|\alpha|=k+1} c_\alpha \int_0^T \|D^\alpha u_k\|_2^2\,dt + \sum_{|\alpha|=0}^{k} a_\alpha \int_0^T \|D^\alpha u_k\|_p^p\,dt \le c_2, \tag{3.1}$$

and

$$\int_Q g_k(t,x,u_k)u_k\,dx\,dt \le c_2. \tag{3.2}$$

In the sequel $c_2, c_3, \dots$ designate arbitrary constants not depending on $k$.

From the first equality in $(\mathrm{P}_k)$ and estimates (3.1) and (3.2), we remark that

$$\frac{\partial u_k}{\partial t} \in L^{p'}(0,T,W_0^{-k+1,p'}(\Omega)).$$

In addition, for any $v \in L^p(0,T,W_0^\infty(a_\alpha,p))$, the following equality is valid

$$\int_0^T \left| \left\langle \frac{\partial u_k}{\partial t}, v \right\rangle \right| dt \le Q_1 + Q_2 + Q_3, \tag{3.3}$$

where

$$Q_1 = \sum_{|\alpha|=0}^{k} a_\alpha \int_0^T\int_\Omega |f_\alpha D^\alpha v|dx\,dt, \qquad Q_2 = \int_0^T\int_\Omega |g_k(t,x,u_k)v|\,dx\,dt,$$

$$Q_3 = \sum_{|\alpha|=k+1} c_\alpha \int_0^T\int_\Omega |D^\alpha u_k||D^\alpha v|\,dx\,dt + \sum_{|\alpha|=0}^{k} a_\alpha \int_0^T\int_\Omega |D^\alpha u_k|^{p-1}|D^\alpha v|\,dx\,dt.$$

Regarding the quantity $Q_1$, one has

$$\begin{aligned} Q_1 &\le \Big( \sum_{|\alpha|=0}^{k} a_\alpha \int_0^T \|f_\alpha\|_{p'}^{p'}\,dt \Big)^{1/p'} \Big( \sum_{|\alpha|=0}^{k} a_\alpha \int_0^T \|D^\alpha v\|_p^p\,dt \Big)^{1/p} \\ &\le (\rho'(f))^{1/p'} \|v\|_{p,\infty}, \end{aligned}$$

and so

$$Q_1 \le c_3\,\|v\|_{p,\infty}. \tag{3.4}$$

Concerning $Q_2$. we have

$$\begin{aligned} Q_2 &\le \int_0^T \int_\Omega (|u_k|^{p-1}|v| + |v|)\,dx\,dt \\ &\le \int_0^T \|u_k\|_p^{p-1}\|v\|_p\,dt + c_4 \Big( \int_0^T \|v\|_p^p\,dt \Big)^{1/p} \\ &\le \Big( \int_0^T \|u_k\|_p^p\,dt \Big)^{1/p'} \Big( \int_0^T \|v\|_p^p\,dt \Big)^{1/p} + c_4 \Big( \int_0^T \|v\|_p^p\,dt \Big)^{1/p} \\ &\le (c_2 + c_4)\|v\|_{p,\infty}, \end{aligned}$$

where $c_2$ is the constant of the estimate (3.1). Then one gets

$$Q_2 \le c_5 \, \|v\|_{p,\infty}. \tag{3.5}$$

Moreover, for the last term $Q_3$, one has $Q_3 = J_1 + J_2$, where

$$\begin{aligned}
J_1 &= \sum_{|\alpha|=k+1} c_\alpha \int_0^T \int_\Omega |D^\alpha u_k||D^\alpha v| \, dx \, dt \\
&\le \bigg( \sum_{|\alpha|=k+1} c_\alpha \int_0^T \|D^\alpha u_k\|_2^2 \, dt \bigg)^{1/2} \bigg( \sum_{|\alpha|=k+1} c_\alpha \int_0^T \|D^\alpha v\|_2^2 \, dt \bigg)^{1/2} \\
&\le (c_2)^{1/2} \|v\|_{p,\infty}, \\
J_2 &= \sum_{|\alpha|=0}^{k} a_\alpha \int_0^T \int_\Omega |D^\alpha u_k|^{p-1} |D^\alpha v| \, dx \, dt \\
&\le \bigg( \sum_{|\alpha|=0}^{k} a_\alpha \int_0^T \|D^\alpha u_k\|_p^p \, dt \bigg)^{1/p'} \bigg( \sum_{|\alpha|=0}^{k} a_\alpha \int_0^T \|D^\alpha v\|_p^p \, dt \bigg)^{1/p} \\
&\le (c_2)^{1/p'} \|v\|_{p,\infty}.
\end{aligned}$$

Then one deduces that

$$Q_3 \le c_6 \|v\|_{p,\,\infty}. \tag{3.6}$$

Combining (3.3)–(3.6), it follows that

$$\int_0^T \left| \left\langle \frac{\partial u_k}{\partial t}, v \right\rangle \right| dt \le c_7 \|v\|_{p,\infty}. \tag{3.7}$$

This implies that

$$\left\| \frac{\partial u_k}{\partial t} \right\|_{p' \, -\infty} \le c_8, \tag{3.8}$$

i.e. the derivatives $\frac{\partial u_k}{\partial t}$ form a bounded set in the space $L^{p'}(0, T, W^{-\infty}(a_\alpha, p'))$.

Now, estimates (3.1) and (3.8) permit us to apply the well known lemma of compactness (see J.L. Lions [13]).

Let $B_0$, $B$ and $B_1$ be Banach spaces. Let us set

$$Y = \{u : u \in L^{p_0}(0, T, B_o), \ u' \in L^{p_1}(0, T, B_1)\}$$

where $p_0 > 1$, $p_1 > 1$ are reals numbers.

Lemma 3.3 (cf. [7]). *Let the imbeddings $B_0 \subset B \subset B_1$ hold; moreover, let the imbedding $B_0 \subset B$ be compact. Then $Y \subset L^{p_0}(0, T, B)$ and this imbedding is compact.*

In order to apply this lemma, define

$$B_0 = W^{S+1}(a_\alpha, p) = \Big\{ u(x) : \sum_{|\alpha|=0}^{S} a_\alpha \|D^\alpha u\|_p^p < \infty \Big\},$$
$$B = W^S(a_\alpha, p), \quad B_1 = W^{-\infty}(a_\alpha, p'); \quad p_0 = p, \quad p_1 = p',$$

where $S \geq 0$ is arbitrary and $p' = p/(p-1)$.

Then in view of estimates (3.1) and (3.8), we deduce that the family $u_k$ of solutions of problems $(\mathrm{P}_k)$ is compact in the space $L^p(0, T, W^S(a_\alpha, p))$, (where $S$ is arbitrary). Consequently, by similar argument as in the elliptic case (using the diagonal process), see [2] or [7], one gets that the sequence $u_k$ converges strongly together with all derivatives $D^\omega u_k$ to a function $u \in L^p(0, T, W_0^\infty(a_\alpha, p))$ in the space $L^p(Q)$.

Let now $m > 0$ fixed, $E$ a measurable subset of $Q$ and $\varepsilon > 0$, we have

$$\begin{aligned}
&\int_E g_k(t, x, u_k)\, dx\, dt \\
&\leq \int_{E\cap\{|u_k|\leq m\}} g_k(t, x, u_k)\, dx\, dt + \frac{1}{m}\int_{E\cap\{|u_k|>m\}} g_k(t, x, u_k) u_k\, dx\, dt \\
&\leq \int_{E\cap\{|u_k|\leq m\}} (|u_k|^{p-1} + 1)\, dx\, dt + \frac{1}{m}\int_Q g_k(t, x, u_k) u_k\, dx\, dt \\
&\leq (|m|^{p-1} + 1)|E| + \frac{c_2}{m},
\end{aligned}$$

where $c_2$ is the constant of (3.2) which is independent of $k$.

For $|E|$ sufficiently small and $c_2/m < \varepsilon/2$, we obtain

$$\int_E g_k(t, x, u_k)\, dx\, dt \leq \varepsilon.$$

Using Vitali's theorem we get

$$g_k(x, t, u_k) \to g(x, t, u) \quad \text{in } L^1(Q). \tag{3.9}$$

On the other hand, in view of Fatou's lemma and (3.2), we obtain

$$\int_Q g(x, t, u) u\, ds \leq \lim_{k\to+\infty} \int_Q g_k(x, t, u_k) u_k\, ds \leq c_2,$$

this implies that

$$g(x, t, u)u \in L^1(Q). \tag{3.10}$$

Now, we shall prove that

$$\lim_{k\to+\infty} \int_0^T \langle A_{2k+2}(u_k), v\rangle\, dt = \int_0^T \langle A(u), v\rangle\, dt$$

for all $v \in L^p(0,T,W_0^\infty(a_\alpha,p))$. Indeed, let $k_0$ be a fixed number sufficiently large and let $v \in L^p(0,T,W_0^\infty(a_\alpha,p))$. Set

$$\int_0^T \langle A(u) - A_{2k+2}(u_k), v\rangle \, dt = I_1 + I_2 + I_3,$$

where

$$\begin{aligned}
I_1 &= \sum_{|\alpha|=0}^{k_0} \int_0^T \langle A_\alpha(t,x,D^\gamma u) - A_\alpha(t,x,D^\gamma u_k), D^\alpha v\rangle \, dt,\\
I_2 &= \sum_{|\alpha|=k_0+1}^{\infty} \int_0^T \langle A_\alpha(t,x,D^\gamma u), D^\alpha v\rangle \, dt,\\
I_3 &= -\sum_{|\alpha|=k_0+1}^{k} \int_0^T \langle A_\alpha(t,x,D^\gamma u_k), D^\alpha v\rangle - \sum_{|\alpha|=k+1} c_\alpha \langle D^\alpha u, D^\alpha v\rangle \, dt,
\end{aligned}$$

or in another form,

$$I_3 = -\sum_{|\alpha|=k_0+1}^{k+1} \int_0^T \langle A_\alpha(t,x,D^\gamma u_k), D^\alpha v\rangle \, dt,$$

with $A_\alpha(t,x,\xi_\gamma) = c_\alpha \xi_\alpha$ and $c_\alpha \geq 0$ for $|\alpha| = k+1$, ($c_\alpha$ are constants given in Lemma 3.2).

The aim is to prove that $I_1$, $I_2$ and $I_3$ tend to 0. Indeed, on one hand, since $A(t,x,\xi_\gamma)$ is of Carathéodory type, then $I_1 \to 0$, and the term $I_2$ is the remainder of a convergence series, hence $I_2 \to 0$. On the other hand, for all $\varepsilon > 0$, there holds $k(\varepsilon) > 0$ (see [5, p. 56]) such that

$$\begin{aligned}
&\bigg| \sum_{|\alpha|=k_0+1}^{k+1} \int_0^T \langle A_\alpha(t,x,D^\gamma u_k), D^\alpha v\rangle \, dt \bigg|\\
&\qquad \leq \sum_{|\alpha|=k_0+1}^{k+1} \int_0^T |\langle A_\alpha(t,x,D^\gamma u_k), D^\alpha v\rangle| \, dt\\
&\qquad \leq c_0 \sum_{|\alpha|=k_0+1}^{k+1} a_\alpha \int_0^T \int_\Omega |D^\alpha u_k|^{p-1} |D^\alpha v| \, dx \, dt\\
&\qquad \leq c_0 \sum_{|\alpha|=k_0+1}^{k+1} a_\alpha \int_0^T \|D^\alpha u_k\|_p^{p-1} \|D^\alpha v\|_p \, dt\\
&\qquad \leq \varepsilon c_0 \sum_{|\alpha|=k_0+1}^{k+1} a_\alpha \int_0^T \|D^\alpha u_k\|_p^p \, dt + c_0 k(\varepsilon) \sum_{|\alpha|=k_0+1}^{k+1} a_\alpha \int_0^T \|D^\alpha v\|_p^p \, dt\\
&\qquad \leq \varepsilon c_0 c_2 + c_0 k(\varepsilon) \sum_{|\alpha|=k_0+1}^{\infty} a_\alpha \int_0^T \|D^\alpha v\|_p^p \, dt,
\end{aligned}$$

where $c_2$ is the constant given in the estimate (3.1). Moreover, the term

$$\sum_{|\alpha|=k_0+1}^{\infty} a_\alpha \int_0^T \|D^\alpha v\|_p^p \, dt$$

is the remainder of a convergent series, therefore $I_3 \to 0$ holds. Finally, we conclude that

$$\int_0^T \langle A_{2k+2}(u_k), v\rangle \, dt \to \int_0^T \langle A(u), v\rangle \, dt \tag{3.11}$$

for all $v \in L^p(0, T, W_0^\infty(a_\alpha, p))$. Moreover, it is clear that

$$\int_0^T \langle f_k, v\rangle \, dt \to \int_0^T \langle f, v\rangle \, dt$$

as $k \to +\infty$. Consequently, by passing to the limit in ($\mathrm{P}_k$), we obtain

$$\int_0^T \left\langle \frac{\partial u}{\partial t}, v \right\rangle dt + \int_0^T \langle A(u), v\rangle \, dt + \int_Q g(t, x, u) v \, dx \, dt = \int_0^T \langle f, v\rangle \, dt,$$

for all $v \in L^p(0, T, W_0^\infty(a_\alpha, p))$. This archived the proof. □

## 4. Example

The following example of an operator of infinite order is closely related to the one used in [8].

Let us consider the operator:

$$Au = \sum_{|\alpha|=0}^{\infty} (-1)^\alpha D^\alpha (a_\alpha |D^\alpha u|^{p-2} D^\alpha u)$$

where $a_\alpha \geq 0$ is a sequence of numbers, $p > 1$ is a number such that the space $W^\infty(a_\alpha, p)(\Omega)$ is not trivial (for example, if $a_\alpha = [(2\alpha)!]^{-p}$, $p > 1$ and dim $\Omega = 1$), then the conditions ($\mathrm{A}_1$)–($\mathrm{A}_3$) are satisfied. As regards to a function $g$ that satisfies the condition (G), let us consider

$$g(t, x, s) = s|s|^r h(x), \quad \text{with } r > 0,$$

where $h \in L^1(\Omega)$, $h(x) \geq 0$, almost everywhere. Consequently, for the described the nonlinear term $g$, the existence result of such a problem of type (P) follows immediately from Theorem 3.1.

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