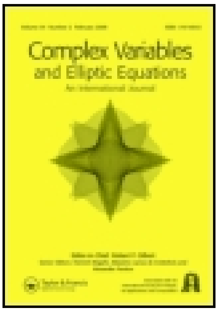

Complex Variables and Elliptic Equations: An International Journal
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/gcov20

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Published online: 14 Feb 2014.

Complex Variables and Elliptic Equations, 2014
http://dx.doi.org/10.1080/17476933.2013.854344

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Communicated by P. Pucci

(*Received 16 April 2013; accepted 8 October 2013*)

In this work, we are interested in the existence of solutions for strongly anisotropic non-linear problems with non-standard growth conditions in the framework of Sobolev spaces of infinite order with variable exponents.

Keywords: strongly non-linear elliptic equations of infinite order; monotonicity condition; sign condition

AMS Subject Classifications: 35J30; 35J60; 35J70

1. Introduction

The study of problems of elliptic equations and variational problems with non-standard growth conditions has attracted more interest in recent years and many papers have appeared. The aim of this paper is to show the existence of solutions of the problem as the following model example:

$$\sum_{|\alpha|=0}^{\infty}(-1)^{|\alpha|}D^{\alpha}\left(a_{\alpha}|D^{\alpha}u|^{p_{\alpha}(x)-2}D^{\alpha}u\right)+u|u|^{r+1}h(x)=f \quad in \;\; \Omega, \tag{1.1}$$

where $\Omega \subset I\!R^N$ is a bounded domain, $a_\alpha \geq 0$ are real numbers, $p_\alpha(.)$ are continuous functions on $\overline{\Omega}$, such that $p_\alpha(x) > 1$ for any $x \in \overline{\Omega}$ and for any multi-indices α, and where $r > 0$ is a real number, $h \in L^1(\Omega)$ with $h(x) \geq 0$ a.e $x \in \Omega$. Note that in the particular case when $p_\alpha(x) = p_\alpha$ for any $x \in \overline{\Omega}$ and any multi-indices α, we refer to the recent works [1] and [2] where the authors proved the existence of solutions in the setting of anisotropic Sobolev spaces of infinite order.

In order to prove the existence of solutions of the elliptic problem of infinite order (1.1), we will first study in Section 3 the same problem but in finite order, by considering the differential operator of order $2m + 2$ defined by

$$A_{2m+2}(u)=\sum_{|\alpha|=0}^{m}(-1)^{|\alpha|}D^{\alpha}\left(a_{\alpha}|D^{\alpha}u|^{p_{\alpha}(x)-2}D^{\alpha}u\right), \tag{1.2}$$

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here $m \geq 1$ is a positive integer. This operator is a natural generalization of the anisotropic operator considered by authors in many works (see e.g. [1–3]), defined as

$$B_{2m+2}(u) = \sum_{|\alpha|=0}^{m} (-1)^{|\alpha|} D^{\alpha}\left(a_{\alpha}|D^{\alpha}u|^{p_{\alpha}-2}D^{\alpha}u\right), \tag{1.3}$$

where $p_{\alpha}(x) = p_{\alpha} > 1$ for all multi-indices $|\alpha| \leq m$. The study of non-linear elliptic equations involving operators type like (1.3) is based on the theory of standard anisotropic Sobolev spaces $W_0^{m,\vec{p}}(\Omega)$ to find weak solutions; recall here that $\vec{p} = \{p_{\alpha}, |\alpha| \leq m\}$. In the case of non-homogeneous operators of kind (1.2), the natural setting for this approach is the use of the variable exponent anisotropic Sobolev spaces. The basic idea is to replace the Lebesgue spaces $L^{p_{\alpha}}(\Omega)$ by more general spaces $L^{p_{\alpha}(x)}(\Omega)$ called variable exponent Lebesgue spaces. In the isotropic case the spaces $L^{p(x)}(\Omega)$ and $W_0^{m,p(x)}(\Omega)$ were thoroughly studied in the monograph by Musielak [4] and the papers by Edmunds et al. [5–7], Mihăilescu and Râdulescu [8], Samko and Vakulov [9] and Diening et al. [10]. Let us mention also, in this direction, the interesting work of Harjulehto et al. [11].

Variable isotropic Sobolev spaces have been used in the last decades to model various phenomena. A major application which uses non-homogeneous operators is related to the modelling of electrorheological fluids, due in the first to Willis Winslow in 1949. For a general account of the underlying physics consult Halsey [12] and for some technical applications Pfeiffer et al. [13]. Electrorheological fluids have been used in robotics and space technology, mainly in the USA, for instance in NASA laboratories. For more information on properties, modelling and the application of variable exponent spaces to these fluids we refer to Acerbi and Mingione [14], Alves and Souto [15], Chabrowski and Fu [16], and Diening [17].

In this paper, the anisotropic Sobolev space of finite order $W_0^{m,\vec{p}}(\Omega)$ or infinite order $W_0^{\infty}(a_{\alpha}, p_{\alpha})(\Omega)$ (see [1,18]) are not adequate to study the non-linear problems of type (1.1) and the problems of finite order involving operators such as (1.2). This leads us to seek weak solutions for our problems in a more general variable anisotropic Sobolev space of finite and infinite order, which will be introduced in the next section of this paper. Let us mention here that the problem of infinite order of type (1.1) with multiple anisotropic variable exponents has not been studied yet, and the present paper may be considered as a first contribution in this direction.

2. Preliminaries

We recall in this section some definitions and basic properties of the variable exponent Lebesgue Sobolev spaces $L^{p(x)}(\Omega)$, where Ω is a bounded subset of $I\!R^N$.

Set

$$C_+(\overline{\Omega}) = \{h \in C(\overline{\Omega}) : \min_{x\in\overline{\Omega}} h(x) > 1\},$$

for any $h \in C_+(\overline{\Omega})$. We define

$$h^+ = \sup_{x\in\Omega} h(x) \qquad \text{and} \qquad h^- = \inf_{x\in\Omega} h(x).$$

For any $p \in C_+(\overline{\Omega})$, we introduce the variable exponent Lebesgue space

$$L^{p(x)} = \left\{u : u \text{ is a measurable real-valued function such that} \int_{\Omega} |u(x)|^{p(x)}\,dx < \infty\right\},$$

endowed with the so-called Luxemburg norm

$$|u|_{p(x)} = inf\left\{\mu > 0 : \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \le 1\right\},$$

which is a separable and reflexive Banach space. For basic properties of the variable exponent Lebesgue spaces we refer to [19].

LEMMA 2.1 (see Fan and Zhao [20] and Zhao et al. [21])

(1) *The space* $(L^{p(x)}(\Omega), |u|_{p(x)})$ *is a separable, uniform convex Banach space, and its conjugate space is* $L^{q(x)}(\Omega)$, *where* $\frac{1}{q(x)} + \frac{1}{p(x)} = 1$. *For any* $u \in L^{p(x)}(\Omega)$ *and* $v \in L^{q(x)}(\Omega)$, *we have*

$$\left|\int_\Omega uv\,dx\right| \le \left(\frac{1}{p^-} + \frac{1}{q^-}\right)|u|_{p(x)}|v|_{q(x)}.$$

(2) *If* $p_1, p_2 \in C_+(\overline{\Omega})$, $p_1(x) \le p_2(x)$ *for any* $x \in \overline{\Omega}$, *then*

$$L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega).$$

and the imbedding is continuous.

LEMMA 2.2 (see Fan and Zhao [20] and Zhao et al. [21])
If we denote

$$\rho(u) = \int_\Omega |u|^{p(x)} dx \qquad \forall u \in L^{p(x)},$$

then

(1) $|u|_{p(x)} < 1\ (= 1; > 1) \Leftrightarrow \rho(u) < 1\ (= 1; > 1)$;
(2) $|u|_{p(x)} > 1 \Rightarrow |u|_{p(x)}^{p^-} \le \rho(u) \le |u|_{p(x)}^{p^+}$;
(3) $|u|_{p(x)} < 1 \Rightarrow |u|_{p(x)}^{p^-} \ge \rho(u) \ge |u|_{p(x)}^{p^+}$;
(4) $|u|_{p(x)} \to 0 \Leftrightarrow \rho(u) \to 0$; $\quad |u|_{p(x)} \to \infty \Leftrightarrow \rho(u) \to \infty$.

LEMMA 2.3 (see Fan and Zhao [20] and Zhao et al. [21])
If $u,\ u_n \in L^{p(x)}(\Omega)$, $n = 0, 1, 2, \ldots$, *then the following statements are equivalent each other:*

(1) $\lim_{n\to\infty} |u_n - u|_{p(x)} = 0$;
(2) $\lim_{n\to\infty} \rho(u_n - u) = 0$;
(3) $u_n \to u$ *in measure in* Ω *and* $\lim_{n\to\infty} \rho(u_n) = \rho(u)$.

Finally, we introduce a natural generalization of the variable exponent Sobolev space $W_0^{m,p(x)}(\Omega)$ that will enable us to study with sufficient accuracy anisotropic problem in Section 3. For this purpose, let us denote by $\vec{p}(x)$ the vectorial function

$$\vec{p}(x) = \{p_\alpha(x),\ |\alpha| \le m\},$$

where m is a positive integer such that $m \geq 1$ and $p_\alpha(.) \in C_+(\overline{\Omega})$ for all multi-indices α such that $|\alpha| \leq m$.

We denote by $C_0^\infty(\Omega)$ the space of all functions with compact support in Ω with continuous derivatives of arbitrary order. We define $W_0^{m,\vec{p}(x)}(\Omega)$, the anisotropic variable exponent Sobolev space, as the closure of $C_0^\infty(\Omega)$ with respect the norm

$$\|u\|_{m,\vec{p}(x)} = \sum_{|\alpha|=0}^{m} |D^\alpha u|_{p_\alpha(x)}.$$

In the case when $p_\alpha(x) \in C_+(\overline{\Omega})$ are constant functions for any $|\alpha| \leq m$, the resulting anisotropic space is denoted by $W_0^{m,\vec{p}}(\Omega)$. Such spaces were developed and considered by authors in [1–3] in the study of some anisotropic strongly non-linear equations. It was proved that $W_0^{m,\vec{p}}(\Omega)$ is a reflexive Banach space for any $p_\alpha > 1$ for all multi-indice $|\alpha| \leq m$. This result can be easily extended to $W_0^{m,\vec{p}(x)}(\Omega)$. In fact, the following lemma follows

Lemma 2.4 *The space* $(W_0^{m,\vec{p}(x)}(\Omega), \|.\|_{m,\vec{p}(x)})$ *is a banach and reflexive space.*

Indeed, as in [22], let

$$X = \prod_{|\alpha|=0}^{m} L^{p_\alpha(x)}(\Omega),$$

and considering the operator

$$T : W_0^{m,\vec{p}(x)}(\Omega) \longrightarrow X,$$

defined by $T(u) = \{D^\alpha u,\ |\alpha| \leq m\}$, for all $u \in W_0^{m,\vec{p}(x)}(\Omega)$. It is clear that $W_0^{m,\vec{p}(x)}(\Omega)$ and X are isometric by T, since

$$\|Tu\|_X = \sum_{|\alpha|=0}^{m} |D^\alpha u|_{p_\alpha(x)} = \|u\|_{m,\vec{p}(x)}.$$

Thus, $T(W_0^{m,\vec{p}(x)}(\Omega))$ is a closed subset of X, which is a reflexive Banach space. By proposition $III.17$ in [23], it follows that $T(W_0^{m,\vec{p}(x)}(\Omega))$ is reflexive and consequently $W_0^{m,\vec{p}(x)}(\Omega)$ is also a reflexive Banach space. On the other hand, in order to facilitate the manipulation of the space $W_0^{m,\vec{p}(x)}(\Omega)$, we introduce p_+^+ and p_-^- as

$$p_+^+ = max\{p_\alpha^+(x),\ |\alpha| \leq m\}, \qquad p_-^- = min\{p_\alpha^-(x),\ |\alpha| \leq m\}.$$

Lemma 2.5 *Let* Ω *be a bounded open subset of* $I\!R^N$.
If $mp_-^- > N$, *then* $W_0^{m,\vec{p}(x)}(\Omega) \subset L^\infty(\Omega) \cap C^k(\overline{\Omega})$ *where* $k = E\left(m - \frac{N}{p_-^-}\right)$.
Moreover, the embedding is compact.

The proof follows immediately from the corresponding embedding theorems in the isotropic case by using the fact that $W_0^{m,\vec{p}(x)}(\Omega) \subset W_0^{m,p_-^-}(\Omega)$.

Now, let $a_\alpha \geq 0$ be a real numbers for multi-indices α. The variable exponent Sobolev space of infinite order is the functional space defined by

$$W^{\infty}(a_\alpha, p_\alpha(x))(\Omega) = \left\{u \in C^{\infty}(\Omega) : \sigma(u) = \sum_{|\alpha|=0}^{\infty} a_\alpha |D^\alpha u|_{p_\alpha(x)}^{p_\alpha^+} < \infty\right\}.$$

Since we are dealing with the Dirichlet problem in this paper, we shall use the functional space $W_0^{\infty}(a_\alpha, p_\alpha(x))(\Omega)$ defined by

$$W_0^{\infty}(a_\alpha, p_\alpha(x))(\Omega) = \left\{u \in C_0^{\infty}(\Omega) : \sigma(u) = \sum_{|\alpha|=0}^{\infty} a_\alpha |D^\alpha u|_{p_\alpha(x)}^{p_\alpha^+} < \infty\right\}.$$

In contrast with the finite order Sobolev space, the very first question, which arises in the study of the spaces $W_0^{\infty}(a_\alpha, p_\alpha(x))(\Omega)$, is the question of their non-triviality (or non-emptiness), i.e. the question of the existence of a function u such that $\sigma(u) < \infty$.

Definition 2.1 (Dubinskii [18]) The space $W_0^{\infty}(a_\alpha, p_\alpha(x))(\Omega)$ is called non-trivial space if it contains at least one function which not identically equal to zero, i.e. there is a function $u \in C_0^{\infty}(\Omega)$ such that $\sigma(u) < \infty$.

It turns out that the answer of this question depends not only on the given parameters a_α, p_α of the spaces $W^{\infty}(a_\alpha, p_\alpha(x))(\Omega)$, but also on the domain Ω.

The dual space of $W_0^{\infty}(a_\alpha, p_\alpha(x))(\Omega)$ is defined as follows

$$W^{-\infty}(a_\alpha, p'_\alpha(x))(\Omega) = \left\{h : h = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha D^\alpha h_\alpha,\ \sigma'(h) = \sum_{|\alpha|=0}^{\infty} a_\alpha |h_\alpha|_{p'_\alpha(x)}^{p'^+_\alpha} < \infty\right\},$$

where $h_\alpha \in L^{p'_\alpha(x)}(\Omega)$ and p'_α is the conjugate of p_α, i.e. $p'_\alpha = \frac{p_\alpha}{p_\alpha - 1}$.

By definition, the duality pairing between $W_0^{\infty}(a_\alpha, p_\alpha(x))(\Omega)$ and its dual space $W^{-\infty}(a_\alpha, p'_\alpha(x))(\Omega)$ is given by the relation

$$\langle h, v\rangle = \sum_{|\alpha|=0}^{\infty} a_\alpha \int_\Omega h_\alpha(x)\, D^\alpha v(x)\, dx,$$

which, as it is not difficult to verify, is correct.

In the particular case when $p_\alpha(x) = p_\alpha$ for any multi-indices α, the Sobolev space of infinite order is defined as

$$W_0^{\infty}(a_\alpha, p_\alpha)(\Omega) = \left\{u \in C_0^{\infty}(\Omega) : \sigma(u) = \sum_{|\alpha|=0}^{\infty} a_\alpha |D^\alpha u|_{p_\alpha}^{r_\alpha} < \infty\right\}.$$

$a_\alpha \geq 0$, $p_\alpha > 1$ and $r_\alpha > 1$ are real numbers for all multi-indices α , and $|.|_{p_\alpha}$ is the usual norm in the Lebesgue space $L^{p_\alpha}(\Omega)$, (see [18,24]).

LEMMA 2.6 *For all non-trivial space $W_0^{\infty}(a_\alpha, p_\alpha(x))(\Omega)$, there exists a non-trivial space $W_0^{\infty}(c_\alpha, 2)(\Omega)$ such that $W_0^{\infty}(a_\alpha, p_\alpha(x))(\Omega) \subset W_0^{\infty}(c_\alpha, 2)(\Omega)$.*

The proof follows immediately from the fact that $W_0^\infty(a_\alpha, p_\alpha(x))(\Omega) \subset W_0^\infty(a_\alpha, p_\alpha^-)(\Omega)$. Thus, by using Lemma 2 of [18], we deduce that there exists a non-triavial space $W_0^\infty(c_\alpha, 2)(\Omega)$ such that $W_0^\infty(a_\alpha, p_\alpha^-)(\Omega) \subset W_0^\infty(c_\alpha, 2)(\Omega)$.

3. Main result

In this section, we formulate and prove the main result of this article.

3.1. *Strongly non-linear problem of finite order*

Let A be the non-linear operator of order $2m$ defined as

$$Au = \sum_{|\alpha|=0}^{m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, D^\gamma u), \qquad |\gamma| \le |\alpha|,$$

where $A_\alpha : \Omega \times I\!R^{\lambda_\alpha} \to I\!R$ is a real function and λ_α is the number of multi-indices γ such that $|\gamma| \le |\alpha|$. Consider the following strongly non-linear problem with Dirichlet conditions:

$$Au + g(x, u) = f \quad \text{in } \Omega.$$

Here, $g : \Omega \times I\!R \to I\!R$ is a measurable function and $f \in W^{-m,\vec{p}'(x)}(\Omega)$. Note that to deal with the Dirichlet problem, we use the anisotropic Sobolev space with variable exponent $W_0^{m,\vec{p}(x)}(\Omega)$ defined in Section 2.

We start by stating the following assumptions:

(H) $A : W_0^{m,\vec{p}(x)}(\Omega) \longrightarrow W^{-m,\vec{p}'(x)}(\Omega)$ is a bounded operator, pseudo-monotone and coercive, i.e.

$$\lim_{\|u\|_{m,\vec{p}(x)} \to +\infty} \frac{< Au, u >}{\|u\|_{m,\vec{p}(x)}} = +\infty.$$

(G) $g : \Omega \times I\!R \longrightarrow I\!R$ is a carathéodory function satisfying

$$\sup_{|u|<s} |g(x, u)| \le h_s(x),$$

for a.e. $x \in \Omega$, all $s > 0$ and some function $h_s \in L^1(\Omega)$. We assume also the "sign condition" $g(x, u)u \ge 0$, for a.e. $x \in \Omega$ and for all $u \in I\!R$.

THEOREM 3.1 *Let $m \in I\!N^*$ such that $mp_-^- > N$. Suppose (H) and (G) are satisfied. Then for all $f \in W^{-m,\vec{p}'(x)}(\Omega)$, there exists $u \in W_0^{m,\vec{p}(x)}(\Omega)$ such that*

$$\begin{cases} g(x, u) \in L^1(\Omega), \quad g(x, u)u \in L^1(\Omega) \\ \langle Au, v\rangle + \int_\Omega g(x, u)v\,dx = \langle f, v\rangle, \quad \forall v \in W_0^{m,\vec{p}(x)}(\Omega). \end{cases}$$

Proof Set $g_k = T_k g$ and $b_k(u, v) = \displaystyle\int_\Omega g_k(x, u)v\,dx$ for all $u, v \in W_0^{m,\vec{p}(x)}(\Omega)$, where

$$T_k\xi = \begin{cases} \xi & \text{if } |\xi| < k \\ \frac{k\xi}{|\xi|} & \text{if } |\xi| \ge k. \end{cases}$$

$b_k(u, v)$ is well defined since $g_k(x, u)$ is bounded with compact support. Define the operator

$$G_k u : W_0^{m,\vec{p}(x)}(\Omega) \to I\!R$$
$$v \to \int_\Omega g_k(x, u) v \, dx.$$

We show that $A + G_k$ is pseudo-monotone. Indeed let $u_j \rightharpoonup u$ weakly in $W_0^{m,\vec{p}(x)}(\Omega)$ such that

$$(A + G_k) u_j \rightharpoonup y \quad \text{weakly in} \quad W^{-m,\vec{p}'(x)}(\Omega),$$

and

$$\lim_{j\to+\infty} \langle (A + G_k) u_j, u_j - u \rangle \leq 0.$$

Then, there exists a subsequence still denoted by u_j such that $u_j \to u$ a.e. in Ω. Hence,

$$\begin{cases} g_k(x, u_j) \to g_k(x, u) & \text{a.e.} x \in \Omega \\ |g_k(x, u_j)| \leq k) & \text{in } \ L^{(p_0^-)'}(\Omega). \end{cases}$$

By Lebesgue's dominated convergence theorem, we get

$$g_k(x, u_j) \to g_k(x, u) \quad \text{in } L^{(p_0^-)'}(\Omega).$$

It follows that

$$\int_\Omega g_k(x, u_j)(u_j - u) \, dx \to 0.$$

Therefore, we obtain

$$Au_j \to y - G_k u \quad \text{weakly in } \ W^{-m,\vec{p}'(x)}(\Omega),$$

and

$$\lim_{j\to+\infty} \langle Au_j, u_j - u \rangle \leq 0.$$

Since A is pseudo-monotone, we deduce

$$y = Au + G_k u.$$

This implies that $A + G_k$ is pseudo-monotone operator. Further, in view of (H) and (G), we easily deduce that $A + G_k$ is bounded and coercive. The operator $A + G_k$ satisfies Lerray-Lions classical conditions (see [25]), so there exists $u_k \in W_0^{m,\vec{p}(x)}(\Omega)$ solution of the problem

$$Au_k + g_k(x, u_k) = f, \tag{3.1}$$

or in its variational formulation

$$\langle Au_k, v \rangle + \int_\Omega g_k(x, u_k) v \, dx = \int_\Omega f v \, dx \qquad \forall v \in W_0^{m,\vec{p}(x)}(\Omega). \tag{3.2}$$

Now substituting $v = u_k$ in (3.2) and using (H) and (G), we get

$$\|u_k\|_{m,\vec{p}(x)} + \int_\Omega g_k(x, u_k) u_k \, dx < C_1 \tag{3.3}$$

for some constant $C_1 > 0$ independent of k. Using this fact and since A is bounded, we obtain

$$\|Au_k\|_{-m,\vec{p}'(x)} < C_2, \tag{3.4}$$

for some constant $C_2 > 0$ independent of k. Recall that $W_0^{m,\vec{p}(x)}(\Omega)$ is reflexive (see Lemma 2.4), we deduce from (3.3) and (3.4) that

$$u_k \rightharpoonup u \quad \text{weakly in } W_0^{m,\vec{p}(x)}(\Omega),$$

$$Au_k \rightharpoonup \chi \quad \text{weakly in } W^{m,\vec{p}'(x)}(\Omega).$$

This implies that we can extract a subsequence still denoted by u_k such that

$$u_k \to u \text{ almost everywhere in } \Omega,$$

$$g_k(x, u_k) \to g(x, u) \text{ almost everywhere in } \Omega.$$

In view of from Fatou's lemma, we get

$$\int_\Omega g(x, u)u\,dx \le \int_\Omega g_k(x, u_k)u_k\,dx \le C_1,$$

which implies that

$$g(x, u)u \in L^1(\Omega).$$

On the other hand, for all $\delta > 0$ we have

$$\begin{aligned}|g_k(x, u_k)| &\le \sup_{|t|<\delta} |g(x, t)| + \delta^{-1}|g(x, u_k)u_k| \\ &\le h_\delta(x) + \delta^{-1}|g(x, u_k)u_k|.\end{aligned}$$

Set E a measurable subset of Ω and $\varepsilon > 0$, we obtain

$$\int_E |g_k(x, u_k)|\,dx \le \int_E h_\delta(x)\,dx + \delta^{-1}C_3,$$

where C_3 is the constant of (3.3) which is independent of k.

For E sufficiently small and $\delta = \frac{2C_3}{\varepsilon}$, we have

$$\int_E |g_k(x, u_k)|\,dx < \varepsilon.$$

Then by Vitali's theorem, we get

$$g_k(x, u_k) \to g(x, u) \quad \text{in } L^1(\Omega).$$

Hence, it follows that $g(x, u) \in L^1(\Omega)$. Passing to the limit in (3.2), we obtain

$$\langle \chi, v\rangle + \int_\Omega g(x, u)v\,dx = \langle f, v\rangle \quad \forall v \in W_0^{m,\vec{p}(x)}(\Omega) \cap L^\infty(\Omega).$$

In view of Lemma 2.5, the last equality holds true for $v = u$ since $W_0^{m,\vec{p}(x)}(\Omega) \subset L^\infty(\Omega)$ when $mp_-^- > N$. Consequently, we have

$$\langle \chi, v\rangle + \int_\Omega g(x, u)v\,dx = \langle f, v\rangle \quad \forall v \in W_0^{m,\vec{p}(x)}(\Omega).$$

Finally, it remains to show that $\chi = Au$. Indeed Fatou's lemma implies that

$$\limsup_{k\to\infty}\langle Au_k, u_k\rangle \leq \langle f, u\rangle - \int_\Omega g(x,u)u\,dx = \langle \chi, u\rangle.$$

So, we have

$$\limsup_{k\to\infty}\langle Au_k, u_k\rangle \leq \langle \chi, u\rangle,$$

and since A is pseudo-monotone, we get $\chi = Au$. Thus, we conclude that

$$\begin{cases} g(x,u) \in L^1(\Omega),\ \ g(x,u)u \in L^1(\Omega) \\ \langle Au, v\rangle + \int_\Omega g(x,u)v\,dx = \langle f, v\rangle, \quad \forall v \in W_0^{m,\vec{p}(x)}(\Omega). \end{cases}$$

□

3.2. *Strongly non-linear problem of infinite order*

Let A be the operator of infinite order defined as

$$Au = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} D^\alpha A_\alpha(x, D^\gamma u), \qquad |\gamma| \leq |\alpha|,$$

where $A_\alpha : \Omega \times I\!R^{\lambda_\alpha} \to I\!R$ is a real function and λ_α is the number of multi-indices γ such that $|\gamma| \leq |\alpha|$. Consider the following strongly non-linear problem with Dirichlet conditions:

$$Au + g(x,u) = f \quad \text{in } \Omega.$$

Here, $g : \Omega \times I\!R \to I\!R$ is a measurable function and $f \in W^{-\infty}(a_\alpha, p'_\alpha(x))(\Omega)$.
Let us now formulate the assumptions:

(A_1) $A_\alpha(x, \xi_\alpha)$ is a Carathéodory function for all α, $|\gamma| \leq |\alpha|$.

(A_2) For a.e. $x \in \Omega$, all $m \in I\!N^*$, all ξ_γ, η_α, $|\gamma| \leq |\alpha|$ and some constant $c_0 > 0$, we assume that

$$\left|\sum_{|\alpha|=0}^{m} A_\alpha(x, \xi_\gamma)\eta_\alpha\right| \leq c_0 \sum_{|\alpha|=0}^{m} a_\alpha |\xi_\alpha|^{p_\alpha(x)-1}|\eta_\alpha|,$$

where $a_\alpha \geq 0$ are reals numbers and $(p_\alpha(.))_\alpha$ is a bounded sequence of functions in $C_+(\overline{\Omega})$ for all multi-indices α.

(A_3) There exist constants $c_1 > 0, c_2 \geq 0$ such that for all $m \in I\!N^*$, for all ξ_γ, ξ_α; $|\gamma| \leq |\alpha|$, we have

$$\sum_{|\alpha|=0}^{m} A_\alpha(x, \xi_\gamma) \cdot \xi_\alpha \geq c_1 \sum_{|\alpha|=0}^{m} a_\alpha |\xi_\alpha|^{p_\alpha(x)} - c_2.$$

(A_4) The space $W_0^\infty(a_\alpha, p_\alpha(x))(\Omega)$ is non-trivial.

(G_1) The function $g : \Omega \times I\!R \mapsto I\!R$ is of Carathéodory type such that, for all $\delta > 0$,

$$\sup_{|u|<\delta} |g(x,u)| \leq h_\delta(x) \in L^1(\Omega).$$

(G$_2$) We assume the "sign condition" $g(x,u)u \geq 0$, for a.e. $x \in \Omega$ and all $u \in I\!R$.

THEOREM 3.2 *Let us assume the conditions* $(A_1) - (A_4)$, (G_1) *and* (G_2). *Then for all* $f \in W^{-\infty}(a_\alpha, p'_\alpha(x))(\Omega)$, *there exists* $u \in W_0^\infty(a_\alpha, p_\alpha(x))(\Omega)$ *such that*

$$\begin{cases} g(x,u) \in L^1(\Omega), \ g(x,u)u \in L^1(\Omega) \\ \langle Au, v\rangle + \int_\Omega g(x,u)v\,dx = \langle f, v\rangle, \quad \text{for all } v \in W_0^\infty(a_\alpha, p_\alpha(x))(\Omega). \end{cases}$$

Proof In order to get our result, we will deal with the following steps:

(1) We prove the existence of approximate solutions u_m.
(2) We establish the a priori estimates.
(3) We prove that u_m converges to an element $u \in W_0^\infty(a_\alpha, p_\alpha(x))(\Omega)$, and we finally show that u is the solution of our problem.

Step 1 The approximate problem.
Define the operator of order $2m+2$ by

$$A_{2m+2}(u) = \sum_{|\alpha|=m+1} (-1)^{m+1} c_\alpha D^{2\alpha}u + \sum_{|\alpha|=0}^{m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, D^\gamma u), \quad |\gamma| \leq m,$$

where c_α are constants small enough such that they fulfill the conditions of the Lemma 2.6. The operator A_{2m+2} is clearly monotone since the term of higher order of derivation is linear and satisfies the monotonicity condition; this follows from the result of [25]. Moreover from assumptions (A_1), (A_2) and (A_3), we deduce that A_{2m+2} satisfies the growth, the coerciveness and the monotonicity conditions. Hence by Theorem 3.1, there exists an approximate solution u_m of the following problem:

$$(\mathrm{Pb_m}) \begin{cases} g(x,u_m) \in L^1(\Omega), \ g(x,u_m)u_m \in L^1(\Omega) \\ \langle A_{2m+2}(u_m), v\rangle + \int_\Omega g(x,u_m)v\,dx = \langle f_m, v\rangle, \ \forall v \in W_0^{m+1,\vec{p}(x)}(\Omega) \end{cases}$$

with

$$f_m = \sum_{|\alpha|=0}^{m} (-1)^{|\alpha|} a_\alpha D^\alpha f_\alpha, \qquad f_\alpha \in L^{p'_\alpha(x)}(\Omega).$$

Step 2 A priori estimate.
Set $v = u_m$ and using (A_3), (G_2), Lemmas 2.1 and 2.2, we deduce the estimates

$$\sum_{|\alpha|=m+1} c_\alpha |D^\alpha u_m|_2^2 + \sum_{|\alpha|=0}^{m} a_\alpha |D^\alpha u_m|_{p_\alpha}^{\beta_\alpha} \leq K \tag{3.5}$$

and

$$\int_\Omega g(x,u_m)u_m\,dx \leq K \tag{3.6}$$

for some constant $K = K(f) > 0$, with

$$\beta_\alpha = \begin{cases} p_\alpha^+ & \text{if} \quad |D^\alpha u|_{p_\alpha(x)} < 1 \\ p_\alpha^- & \text{if} \quad |D^\alpha u|_{p_\alpha(x)} > 1. \end{cases}$$

From this and since the summation in estimate (3.5) is finite, we can also write

$$\sum_{|\alpha|=m+1} c_\alpha |D^\alpha u_m|_2^2 + \sum_{|\alpha|=0}^{m} a_\alpha |D^\alpha u_m|_{p_\alpha}^{p_\alpha^+} \le K. \tag{3.7}$$

The estimate (3.7) is equivalent to

$$\sum_{|\alpha|=0}^{m+1} a_\alpha |D^\alpha u_m|_{p_\alpha(x)}^{p_\alpha^+} \le K, \tag{3.8}$$

with $a_\alpha = c_\alpha$ and $p_\alpha = 2$ for $|\alpha| = m+1$. Consequently, we have

$$\|u_m\|_{W^{m+1,\vec{p}(x)}} \le K. \tag{3.9}$$

Then via a diagonalization process, there exists a subsequence still, denoted by u_m, which converges uniformly to an element $u \in C_0^\infty(\Omega)$, also for all derivatives there holds $D^\alpha u_m \to D^\alpha u$ (for more details we refer to [1,18]).

Step 3 Convergence of problem (Pb_m).

There exists a solution u_m of problem (Pb_m), $m = 1, 2, \ldots$. Then by passing to the limit, we have

$$\lim_{m\to+\infty} \langle A_{2m+2}(u_m), v\rangle + \lim_{m\to+\infty} \int_\Omega g(x, u_m) v\,dx = \lim_{m\to+\infty} \langle f_m, v\rangle,$$

for $v \in W_0^\infty(a_\alpha, p_\alpha(x))(\Omega)$. It is clear that

$$\lim_{m\to+\infty} \langle f_m, v\rangle = \langle f, v\rangle \qquad \text{for all} \quad v \in W_0^\infty(a_\alpha, p_\alpha(x))(\Omega).$$

Now, we shall prove that

$$\lim_{m\to+\infty} \langle A_{2m+2}(u_m), v\rangle = \langle Au, v\rangle, \qquad \text{for all } v \in W_0^\infty(a_\alpha, p_\alpha(x))(\Omega).$$

In fact, let m_0 be a fix number sufficiently large ($m > m_0$) and let $v \in W_0^\infty(a_\alpha, p_\alpha)(\Omega)$. Set

$$\langle A(u) - A_{2m+2}(u_m), v\rangle = I_1 + I_2 + I_3,$$

where

$$I_1 = \sum_{|\alpha|=0}^{m_0} \langle A_\alpha(x, D^\gamma u) - A_\alpha(x, D^\gamma u_m), D^\alpha v\rangle$$

$$I_2 = \sum_{|\alpha|=m_0+1}^{\infty} \langle A_\alpha(x, D^\gamma u), D^\alpha v\rangle$$

$$I_3 = -\sum_{|\alpha|=m_0+1}^{m} \langle A_\alpha(x, D^\gamma u_m), D^\alpha v\rangle - \sum_{|\alpha|=m+1} c_\alpha \langle D^\alpha u_m, D^\alpha v\rangle,$$

or in another form,

$$I_3 = -\sum_{|\alpha|=m_0+1}^{m+1} \langle A_\alpha(x, D^\gamma u_m), D^\alpha v\rangle.$$

with $A_\alpha(x, \xi_\gamma) = c_\alpha \xi_\alpha$ and $c_\alpha \ge 0$ for $|\alpha| = m+1$.

We will go to the limit as $m \to +\infty$ to prove that I_1, I_2 and I_3 tend to 0. Starting by I_1, we have $I_1 \to 0$ since $A_\alpha(x, \xi_\gamma)$ is of Carathéodory type. The term I_2 is the remainder of a convergent series; hence $I_2 \to 0$. For what concerns I_3, in view of (A_2) and Hölder inequality (Lemma 2.1) we have

$$\begin{aligned}\left|\sum_{|\alpha|=m_0+1}^{m+1} \langle A_\alpha(x, D^\gamma u_m), D^\alpha v\rangle\right| &\leq \sum_{|\alpha|=m_0+1}^{m+1} |\langle A_\alpha(x, D^\gamma u_m), D^\alpha v\rangle| \\ &\leq c_0 \sum_{|\alpha|=m_0+1}^{m+1} a_\alpha \int_\Omega |D^\alpha u_m|^{p_\alpha(x)-1} |D^\alpha v|\, dx \\ &\leq c_0 \sum_{|\alpha|=m_0+1}^{m+1} a_\alpha |\ |D^\alpha u_m|^{p_\alpha(x)-1}|_{p'_\alpha(x)} |D^\alpha v|_{p_\alpha(x)}.\end{aligned}$$

Now, in view of Lemma 2.3, one can get

$$\begin{aligned}|\ |D^\alpha u_m|^{p_\alpha(x)-1}|_{p'_\alpha(x)} &\leq \left(\int_\Omega |D^\alpha u_m|^{(p_\alpha(x)-1)p'_\alpha(x)}\, dx\right)^{\mu_\alpha} \\ &\leq \left(\int_\Omega |D^\alpha u_m|^{p_\alpha(x)}\, dx\right)^{\mu_\alpha} \\ &\leq |D^\alpha u_m|_{p_\alpha(x)}^{\mu_\alpha\ \beta_\alpha} \\ &\leq |D^\alpha u_m|_{p_\alpha(x)}^{p_\alpha^+ - 1},\end{aligned}$$

where μ_α and β_α are real numbers for all multi-indices $|\alpha| \leq m$ defined as

$$\mu_\alpha = \begin{cases} \frac{1}{p'^{+}_\alpha} & \text{if} \quad |\ |D^\alpha u_m|^{p_\alpha(x)-1}|_{p'_\alpha(x)} < 1 \\ \frac{1}{p'^{-}_\alpha} & \text{if} \quad |\ |D^\alpha u_m|^{p_\alpha(x)-1}|_{p'_\alpha(x)} > 1. \end{cases}$$

$$\beta_\alpha = \begin{cases} p_\alpha^+ & \text{if} \quad |D^\alpha u_m|_{p_\alpha(x)} < 1 \\ p_\alpha^- & \text{if} \quad |D^\alpha u_m|_{p_\alpha(x)} > 1. \end{cases}$$

It is very easy to verify that for all multi-indices $|\alpha| \leq m$, on has

$$\mu_\alpha\ \beta_\alpha \leq p_\alpha^+ - 1.$$

Therefore, for all $\varepsilon > 0$, there exists $k(\varepsilon) > 0$ (see [23, p.56]) such that

$$\begin{aligned}\left|\sum_{|\alpha|=m_0+1}^{m+1} \langle A_\alpha(x, D^\gamma u_m), D^\alpha v\rangle\right| &\leq \varepsilon c_0 \sum_{|\alpha|=m_0+1}^{m+1} a_\alpha |D^\alpha u_m|_{p_\alpha(x)}^{p_\alpha^+} + c_0 k(\varepsilon) \sum_{|\alpha|=m_0+1}^{m+1} a_\alpha |D^\alpha v|_{p_\alpha(x)}^{p_\alpha^+} \\ &\leq \varepsilon c_0 K + c_0 k(\varepsilon) \sum_{|\alpha|=m_0+1}^{\infty} a_\alpha |D^\alpha v|_{p_\alpha(x)}^{p_\alpha^+},\end{aligned}$$

where K is the constant given in the estimate (3.8). Since the sequence $(p_\alpha(x))$ is bounded and $\sum_{|\alpha|=m_0+1}^{\infty} a_\alpha |D^\alpha v|_{p_\alpha(x)}^{p_\alpha^+}$ is the remainder of a convergent series, therefore $I_3 \to 0$

holds. Hence, $\langle A_{2m+2}(u_m), v\rangle \to \langle A(u), v\rangle$ as $m \to +\infty$ for all $v \in W_0^{\infty}(a_\alpha, p_\alpha(x))(\Omega)$. It remains to show, for our purposes, that

$$\lim_{m\to+\infty} \int_\Omega g(x, u_m)v\, dx = \int_\Omega g(x, u)v\, dx,$$

for all $v \in W_0^{\infty}(a_\alpha, p_\alpha(x))(\Omega)$. Indeed, we have $u_m \to u$ uniformly in Ω; hence $g(x, u_m) \to g(x, u)$ for a.e. $x \in \Omega$. In view of (3.6), we deduce by Fatou's lemma that

$$\int_\Omega g(x, u)u\, dx \le \lim_{m\to+\infty} \int_\Omega g(x, u_m)u_m\, dx \le K.$$

This implies that $g(x, u)u \in L^1(\Omega)$. On the other hand, let $\delta > 0$, since $|g(x, t)|\delta \le |g(x, t)t|$ and then $|g(x, t)| \le \delta^{-1}|g(x, t)t|$ for $|t| \ge \delta$, we have

$$\begin{aligned} |g(x, u_m)| &\le \sup_{|t|\le\delta} |g(x, t)| + \delta^{-1}|g(x, u_m).u_m| \\ &\le h_\delta(x) + \delta^{-1}|g(x, u_m)u_m|. \end{aligned}$$

It follows that

$$\int_E |g(x, u_m)|\, dx \le \int_E h_\delta(x)\, dx + \delta^{-1}K,$$

for some measurable subset E of Ω and for some $\varepsilon > 0$. Here, K is the constant of (3.2) which is independent of m. For $|E|$ sufficiently small and $\delta = \frac{2K}{\varepsilon}$, we obtain $\int_E |g(x, u_m)|\, dx < \varepsilon$. Then, using Vitali's, we get theorem $g(x, u_m) \to g(x, u)$ in $L^1(\Omega)$. Hence, it follows that $g(x, u) \in L^1(\Omega)$.

By passing to the limit, we obtain

$$\langle Au, v\rangle + \int_\Omega g(x, u)v\, dx = \langle f, v\rangle, \quad \text{for all } v \in W_0^{\infty}(a_\alpha, p_\alpha(x))(\Omega).$$

Consequently,

$$\begin{cases} g(x, u) \in L^1(\Omega),\ g(x, u)u \in L^1(\Omega) \\ \langle Au, v\rangle + \int_\Omega g(x, u)v\, dx = \langle f, v\rangle \quad \text{for all } v \in W_0^{\infty}(a_\alpha, p_\alpha(x))(\Omega) \end{cases}$$

This completes the proof. □

Remark 3.1 Note that the existence result is given with no monotonicity condition on the operator.

Acknowledgements

The research of S. El Manouni is supported by the Arab Fund for Economic and Social Development (AFESD), Kuwait.

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